Integrability of the Egorov hydrodynamic type systems

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Abstract

Integrability criterion for the Egorov hydrodynamic type systems is presented. The general solution by the generalized hodograph method is found. Examples are given. A description of three orthogonal curvilinear coordinate nets is discussed from the viewpoint of reciprocal transformations.

In honour of Sergey Tsarev

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1 Introduction

The theory of integrable hydrodynamic type systems

$$u_t^i = \sum_{j=1}^{N} v_j^i(\mathbf{u}) u_x^j, \qquad i = 1, 2, ..., N \tag{1}$$

was created by B.A. Dubrovin and S.P. Novikov (see [4]) and developed by S.P. Tsarev (see [25]).

An integrability of the hydrodynamic type systems (1) was investigated by S.P. Tsarev in [25] for the distinct characteristic velocities v^i, which are determined by the algebraic system

$$\det \left| v_k^i(\mathbf{u}) - v\delta_k^i \right| = 0.$$

If (1) can be written via the Riemann invariants $r^i(\mathbf{u})$ ($i = 1, 2, ..., N$) in the diagonal form

$$r_t^i = v^i(\mathbf{r}) r_x^i,$$

then the integrability condition (so-called the "semi-Hamiltonian" property) is given by

$$\partial_j \frac{\partial_k v^i}{v^k - v^i} = \partial_k \frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j \neq k,$$

where $\partial_k \equiv \partial / \partial r^k$. Any semi-Hamiltonian hydrodynamic type system (1) possesses an infinite set of conservation laws

$$\partial_t h(\mathbf{u}) = \partial_x p(\mathbf{u}) \tag{2}$$

and an infinite set of commuting flows (i.e. the functions u^k simultaneously are functions of x, t and τ, then the compatibility conditions $(u_\tau^i)_t = (u_t^i)_\tau$ must be fulfilled)

$$u_\tau^i = \sum_{j=1}^{N} w_j^i(\mathbf{u}) u_x^j, \qquad i = 1, 2, ..., N \tag{3}$$

parameterized by N arbitrary functions of a single variable (see [25]). The algebraic system

$$x\delta_k^i + t v_k^i(\mathbf{u}) = w_k^i(\mathbf{u}), \qquad i, k = 1, 2, ..., N. \tag{4}$$

yields (in an implicit form) a general solution for (1). This is the *generalized hodograph* method.

Definition ([19], [20], [23]): *The semi-Hamiltonian hydrodynamic type system* (1) *is said to be the Egorov, if* (1) *has the couple of conservation laws*

$$\partial_t a = \partial_x b, \qquad \partial_t b = \partial_x c. \tag{5}$$

If (1) is the Egorov hydrodynamic type system, then each commuting flow (3) has the similar pair of conservation laws (5) (see details in [19], [20], [23])

$$\partial_\tau a = \partial_x h, \qquad \partial_\tau h = \partial_x f. \tag{6}$$

In this paper we describe a very important class of conservation laws – the Egorov hydrodynamic type systems (see [3], [13], [14], [15], [17], [19], [20], [23], [25], [26]), which have plenty applications in different areas of pure and applied mathematics, physics, biology and chemistry.

In this paper we introduce the ***extended*** *hodograph method* for the Egorov hydrodynamic type systems; we significantly improve the generalized hodograph method (4) for the Egorov hydrodynamic type systems; if the Egorov hydrodynamic type systems possess Hamiltonian structure, then N series of conservation laws and commuting flows can be found iteratively.

The paper is organized in the following order. In the second section the Egorov hydrodynamic type systems are considered with the aid of the differential geometry (conjugate curvilinear coordinate nets). The *Egorov basic set* of conservation laws and commuting flows is introduced. In the third section the generalized hodograph method is adopted for the Egorov hydrodynamic type systems. In the fourth section the extended hodograph method is presented. In the fifth section the integrability criterion of the Egorov hydrodynamic type systems is given. In the sixth section orthogonal curvilinear coordinate nets with symmetric rotation coefficients are considered. In the seventh section the Egorov hydrodynamic type systems possessing a local Hamiltonian structure are investigated. Corresponding associativity equations are derived. In the eighth section reciprocal transformations preserving local Hamiltonian structure are found. In the ninth section the Egorov three orthogonal curvilinear coordinate nets are considered. Transformations connecting the Egorov hydrodynamic type systems associated with local Hamiltonian structures are described. In the tenth section the Egorov hydrodynamic type systems associated with nonlocal Hamiltonian structure are briefly mentioned.

2 *Conjugate* curvilinear coordinate nets

The theory of *conjugate curvilinear coordinate nets* is developed by G. Darboux (see [2]). The linear system

$$\partial_i H_k = \beta_{ik} H_i, \qquad i \neq k \tag{7}$$

is integrable, if (i.e. if $\partial_j(\partial_i H_k) = \partial_i(\partial_j H_k), \quad i \neq j \neq k$)

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \qquad i \neq j \neq k. \tag{8}$$

The functions β_{ik} are said to be *the rotation coefficients of conjugate curvilinear coordinate nets*, the functions H_i are said to be *the Lame coefficients*.

Let us take some solution of the nonlinear PDE system (8). Suppose the general solution H_i of the linear PDE system (7) (parameterized by N arbitrary functions of a single variable) is found. Let us take any two arbitrary solutions $H_{(2)i}$ and $H_{(1)i}$ of (7), then one can construct the integrable hydrodynamic type system (see [25])

$$r^i_{t^2} = \frac{H_{(2)i}}{H_{(1)i}} r^i_{t^1}, \tag{9}$$

where the Riemann invariants r^i are implicit functions of "times" t^1 and t^2, which usually are called as x and t, respectively. The general solution of this hydrodynamic type system is given by the generalized hodograph method (see [25]):

Theorem [25]: *The algebraic system*

$$xH_{(1)i} + tH_{(2)i} = H_i, \qquad i = 1, 2, ..., N \tag{10}$$

yields a general solution (in an implicit form) of the hydrodynamic type system (9).

Thus, the general solution of the nonlinear PDE system (8) together with the general solution of the linear PDE system (7) describe all possible semi-Hamiltonian hydrodynamic type systems (9).

Taking arbitrary solutions $\psi_i^{(\beta)}$ of the *adjoint* linear system (see (7))

$$\partial_i \psi_k = \beta_{ki} \psi_i, \qquad i = 1,\ 2,\ .\ .\ .\ ,\ N, \tag{11}$$

one can construct conservation laws written in the potential form

$$dz^\beta = \sum_\gamma a^\beta_\gamma(\mathbf{r}) dt^\gamma \tag{12}$$

for the hydrodynamic type systems

$$r^i_{t^\beta} = \frac{H_{(\beta)i}}{H_{(\gamma)i}} r^i_{t^\gamma}, \qquad i = 1,\ 2,\ .\ .\ .\ ,\ N, \tag{13}$$

where

$$\partial_i a^\beta_\gamma = \psi_i^{(\beta)} H_{(\gamma)i}. \tag{14}$$

It means, for example, that the hydrodynamic type system (9) has an infinite number of the conservation laws

$$\partial_{t^2} a^\beta_1 = \partial_{t^1} a^\beta_2,$$

parameterized by N arbitrary functions of a single variable.

Let us consider the hydrodynamic type system (9) together with its $M-2$ nontrivial commuting flow (see (13))

$$r^i_{t^m} = \frac{H_{(m)i}}{H_{(1)i}} r^i_{t^1}, \qquad m = 2, 3, ..., N. \tag{15}$$

Then the generalized hodograph method (see (10)) yields the general solution (see [14])

$$xH_{(1)i} + tH_{(2)i} + \sum_{k=3}^{N} t^k H_{(k)i} = H_i, \tag{16}$$

where N Riemann invariants r^i are functions of N independent variables t^k. This is invertible transformation $r^i(t^1, t^2, ..., t^N)$.

In this article we restrict a consideration of the hydrodynamic type systems (9) (see also (13) and (15)) on the symmetric case

$$\beta_{ik} = \beta_{ki}, \qquad i \neq k. \tag{17}$$

Corresponding conjugate curvilinear coordinate nets (8) were introduced by G. Darboux in 1866 and investigated by D.Th. Egorov in 1901 in his thesis (see [5]). It was G. Darboux (see [2]) who proposed to call them (see (8) and (17)) the Egorov curvilinear coordinate systems. From the point of view of integrability properties a remarkable progress was achieved by L. Bianchi in 1915 (see [1]). He found a Bäcklund transformation for this problem and established the permutability property as well as the superposition formula for it in the flat case, i.e. when conjugate curvilinear coordinate net becomes to be orthogonal

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = 0. \tag{18}$$

Namely, the Egorov orthogonal curvilinear coordinate nets were extensively investigated at that time.

If rotation coefficients β_{ik} are symmetric, then the linear problems (7) and (11) coincide, and corresponding conservation laws (see (12) and (14)) can be taken in the symmetric form too

$$d\Omega_\beta = \sum_\gamma a_{\beta\gamma} dt^\gamma, \tag{19}$$

where

$$\partial_i a_{\beta\gamma} = H_{(\beta)i} H_{(\gamma)i}. \tag{20}$$

It means that one can introduce the function Ω determined by its second derivatives

$$a_{\beta\gamma} = \frac{\partial^2 \Omega}{\partial t^\beta \partial t^\gamma}, \qquad \Omega_\beta = \frac{\partial \Omega}{\partial t^\beta}.$$

Then the commuting flows (15) are determined by the unique function Ω. Thus, the hydrodynamic type system (9) has the Egorov couple of conservation laws (5) (see (20)), where

$$\partial_i a_{11} = H^2_{(1)i}, \qquad \partial_i a_{12} = H_{(2)i} H_{(1)i}, \qquad \partial_i a_{22} = H^2_{(2)i}, \tag{21}$$

Definition ([19], [20], [23]): *The conservation law density a_{11} is said to be the* ***potential*** *of the Egorov metric.*

Let us take N particular solutions $H_{(\beta)i}$ of the linear system (7). If the rotation coefficients β_{ik} are symmetric (see (17)), then the Egorov hydrodynamic type systems (15) can be written together in the symmetric potential form (19).

In the next section we improve the generalized hodograph method (see (10), (16)) for the Egorov hydrodynamic type systems.

3 The *generalized* hodograph method

Let us introduce N field variables a_k such that $\partial_i a_k = H_{(k)i} H_{(1)i}$ (i.e. $a_k \equiv a_{k1}$, see (**19**) and (**20**)).

Then the algebraic system (**16**) for the Egorov hydrodynamic type systems (**15**)

$$\partial_{t^k} a_n = \partial_{t^1} a_{kn}(\mathbf{a}) \tag{22}$$

can be written in the form

$$\sum_{k=1}^{N} t^k \partial_i a_k = \partial_i h \qquad \Leftrightarrow \qquad \sum_{k=1}^{N} t^k da_k = dh, \tag{23}$$

where the conservation law density $h(a_1, a_2, ..., a_N)$ is determined by its derivatives $\partial_i h = H_i H_{(1)i}$ (see (**20**)) and parameterized by N arbitrary functions of a single variable. The fluxes $p_k(\mathbf{a})$ of this conservation law (cf. (**2**))

$$\partial_{t^k} h(\mathbf{a}) = \partial_{t^1} p_k(\mathbf{a}) \tag{24}$$

can be found in quadratures

$$dp_k(\mathbf{a}) = \frac{\partial h(\mathbf{a})}{\partial a_n} da_{kn}(\mathbf{a}).$$

Lemma: *The Egorov hydrodynamic type systems* (**22**) *possess an arbitrary commuting flow (see* (**3**) *and* (**6**))

$$\partial_\tau a_1 = \partial_{t^1} h(\mathbf{a}), \qquad \partial_\tau a_k = \partial_{t^1} p_k(\mathbf{a}), \qquad k = 2, 3, ..., N. \tag{25}$$

Proof: can be obtained from the compatibility condition $\partial_{t^k}(\partial_\tau a_1) = \partial_\tau(\partial_{t^k} a_1)$.

Thus, we have the generalized hodograph method *adopted* for the Egorov hydrodynamic type systems.

Theorem: *A general solution of the Egorov hydrodynamic type systems* (**15**) *is given by the algebraic system (in an implicit form, see* (**23**)*)*

$$t^k = \frac{\partial h(\mathbf{a})}{\partial a_k}. \tag{26}$$

Corollary: The function Ω can be expressed explicitly via the field variables a_k and found in quadratures in two steps (see (**19**) and below)

$$d\Omega_k = a_{ks}(\mathbf{a}) d\frac{\partial h(\mathbf{a})}{\partial a_k}, \qquad d\Omega = \Omega_k(\mathbf{a}) d\frac{\partial h(\mathbf{a})}{\partial a_k}.$$

In the next section we introduce the extended hodograph method for integrability of the Egorov hydrodynamic type systems based on this symmetric representation.

4 The *extended* hodograph method

Any two-component system (1) is integrable by the *hodograph* method (see, for instance, [24]). Let us adopt this hodograph method for the Egorov hydrodynamic type systems. The hydrodynamic type system (1) can be written in the conservative form (5). This couple of conservation laws can be written in the potential form

$$dy = adx + bdt, \qquad dz = bdx + c(a,b)dt.$$

Using the Legendre transform $\Phi = ax + bt - y$, $\xi = bx + ct - z$ the above couple of equations can be written in the form

$$d\Phi = xda + tdb, \qquad d\xi = xdb + tdc(a,b).$$

Since $x_b = t_a$ (see the first above equation), then (see the second above equation) the compatibility condition $(\xi_a)_b = (\xi_b)_a$

$$(x + tc_b)_a = (tc_a)_b \tag{27}$$

can be written as the *linear* PDE equation of the second order

$$\Phi_{aa} + c_b\Phi_{ab} = c_a\Phi_{bb}. \tag{28}$$

Thus, the hodograph method is the transformation from the field variables (unknown functions) $a(x,t)$ and $b(x,t)$ of the **quasilinear** system (1) to the new field variables (independent variables) $x(a,b)$ and $t(a,b)$ of the **linear** system (27).

Remark: Since (see (26)) $x = h_a$ and $t = h_b$, then $\Phi \equiv h$ and $\xi \equiv p$. Thus, the above equation (28) coincides with the equation describing conservation law densities $h(a,b)$ of the hydrodynamic type system $a_t = b_x$, $b_t = \partial_x c(a,b)$.

Example: the nonlinear elasticity equation (see, for instance, [18]) is determined by the function $c(a)$. The above linear equation is reducible to the hypergeometric equation if $c(a) = a^n$, $c(a) = \ln a$, $c(a) = e^a$.

Let us consider the couple of three component Egorov hydrodynamic type system

$$\begin{aligned} a_t &= b_x, & a_y &= c_x, \\ b_t &= \partial_x u(a,b,c), & b_y &= \partial_x v(a,b,c), \\ c_t &= \partial_x v(a,b,c), & c_y &= \partial_x w(a,b,c), \end{aligned} \tag{29}$$

which can be written in the potential symmetric form (22)

$$d\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & u & v \\ c & v & w \end{pmatrix} d\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix}, \tag{30}$$

where $y = t^3$.

If functions $a(x,t,y)$, $b(x,t,y)$ and $c(x,t,y)$ are common for both hydrodynamic type systems (29), then the compatibility condition $\partial_t(\partial_y a) = \partial_y(\partial_t a)$ *satisfies identically* and the compatibility condition $\partial_t(\partial_y b) = \partial_y(\partial_t b)$ is *identically coincides* with the compatibility condition $\partial_t(\partial_y c) = \partial_y(\partial_t c)$.

Lemma: *If two hydrodynamic type systems* (**29**) *commute, then the relationship between three coefficients* $u(a,b,c)$, $v(a,b,c)$ *and* $w(a,b,c)$ *is given by*

$$\begin{aligned}
\partial_b \frac{u_a v_b + v_a v_c - u_b v_a}{u_c} &= \partial_a \frac{v_a + v_b v_c}{u_c}, \\
\partial_b \frac{u_c v_b + v_c^2 - u_b v_c - u_a}{u_c} &= \partial_c \frac{v_a + v_b v_c}{u_c}, \\
\partial_a \frac{u_c v_b + v_c^2 - u_b v_c - u_a}{u_c} &= \partial_c \frac{u_a v_b + v_a v_c - u_b v_a}{u_c},
\end{aligned} \tag{31}$$

where

$$dw = \frac{(u_a v_b + v_a v_c - u_b v_a)da + (v_a + v_b v_c)db + (u_c v_b + v_c^2 - u_b v_c - u_a)dc}{u_c}. \tag{32}$$

Under the Legendre transformation $h = at^1 + bt^2 + ct^3 - \xi^1$, $p = bt^1 + ut^2 + vt^3 - \xi^2$, $q = ct^1 + vt^2 + wt^3 - \xi^3$, the above symmetric system (**30**) reduces to the three differentials

$$dh = t^1 da + t^2 db + t^3 dc, \qquad dp = t^1 db + t^2 du + t^3 dv, \qquad dq = t^1 dc + t^2 dv + t^3 dw.$$

Since (see the first above equation and (**26**)) $t^1 = h_a$, $t^2 = h_b$, $t^3 = h_c$, then (see the second and third above equations) the compatibility conditions ($(p_a)_b = (p_b)_a$, $(q_a)_b = (q_b)_a$, etc)

$$\begin{aligned}
(t^2 u_a + t^3 v_a)_b &= (t^1 + t^2 u_b + t^3 v_b)_a, \\
(t^2 u_c + t^3 v_c)_a &= (t^2 u_a + t^3 v_a)_c, \\
(t^2 u_c + t^3 v_c)_b &= (t^1 + t^2 u_b + t^3 v_b)_c
\end{aligned}$$

can be written as the *linear* PDE system (where we use the temporary notation $\Phi = \xi^1$)

$$\begin{aligned}
u_a \Phi_{bb} + v_a \Phi_{bc} &= \Phi_{aa} + u_b \Phi_{ab} + v_b \Phi_{ac}, \\
u_a \Phi_{bc} + v_a \Phi_{cc} &= u_c \Phi_{ab} + v_c \Phi_{ac}, \\
u_c \Phi_{bb} + v_c \Phi_{bc} &= \Phi_{ac} + u_b \Phi_{bc} + v_b \Phi_{cc}.
\end{aligned} \tag{33}$$

Thus, the above described transformation from the field variables (unknown functions) $a(x,t,y)$, $b(x,t,y)$ and $c(x,t,y)$ of the **quasilinear** system (**29**) to the new field variables (independent variables) $x(a,b,c)$, $t(a,b,c)$ and $y(a,b,c)$ of the **linear** system (**33**) is nothing but the *extended hodograph method* for a couple of three component hydrodynamic type systems.

The compatibility conditions $(\Phi_{cc})_b = (\Phi_{bc})_c$, $(\Phi_{cc})_a = (\Phi_{ac})_c$, $(\Phi_{ac})_b = (\Phi_{bc})_a$ of the over-determined system (**33**) are equivalent (**31**).

Remark: Since (see (**26**)) $x = h_a$, $t = h_b$ and $y = h_c$, then (see (**24**)) $\xi^1 \equiv h$, $\xi^1 \equiv p$ and $\xi^1 \equiv q$ (where we use temporary notation $F_1 = p$ and $F_2 = q$). Thus, the above system (**33**) coincides with the system describing conservation law densities $h(a, b, c)$ of both hydrodynamic type systems (**29**).

Obviously, this construction easily can be extended on N component case.

5 *Natural* extra commuting flows. The integrability criterion

Without lost of generality let us restrict our consideration on the first nontrivial three component Egorov hydrodynamic type system written in the conservative form

$$a_t = b_x, \qquad b_t = \partial_x u(a, b, c), \qquad c_t = \partial_x v(a, b, c). \tag{34}$$

If this hydrodynamic type system is semi-Hamiltonian, then it must admit the *natural* commuting flow (cf. (**29**))

$$a_y = c_x, \qquad b_y = \partial_x v(a, b, c), \qquad c_y = \partial_x w(a, b, c). \tag{35}$$

Indeed, since (**34**) is semi-Hamiltonian, then this hydrodynamic type system has an infinite series of conservation laws

$$\partial_t h(a, b, c) = \partial_x p(a, b, c)$$

and commuting flows. Each of them can be written in the form (**25**) (see [23])

$$a_\tau = \partial_x h(a, b, c), \qquad b_\tau = \partial_x p(a, b, c), \qquad c_\tau = \partial_x q(a, b, c), \tag{36}$$

where the commuting flow (**35**) has the conservation law

$$\partial_y h(a, b, c) = \partial_x q(a, b, c).$$

The function $h(a, b, c)$ satisfies the linear PDE system of the first order with variable coefficients. Thus, it is very difficult to find the function $h(a, b, c)$ in general case. However, at least three particular solutions $h(a, b, c)$ are given a priori (see (**34**)); i.e. $h_{(1)} = a$, $h_{(2)} = b$ and $h_{(3)} = c$. The first choice is trivial, the second choice is given by (**34**). Thus, we can try to reconstruct an extra commuting flow, where the first conservation law is

$$a_y = c_x.$$

Moreover, the compatibility condition $\partial_y(\partial_t a) = \partial_t(\partial_y a)$ leads to the next conservation law

$$b_y = \partial_x v(a, b, c).$$

Finally, the compatibility condition $\partial_y(\partial_t b) = \partial_t(\partial_y b)$ leads to the criterion of an integrability for the three component Egorov hydrodynamic type systems (**34**).

Criterion of integrability: *The Egorov hydrodynamic type system* (**34**) *has the commuting flow iff the function* w *can be found in quadratures* (**32**).

The compatibility condition $\partial_y(\partial_t c) = \partial_t(\partial_y c)$ leads to the same result. Thus, if the compatibility conditions are fulfilled, then the Egorov hydrodynamic type system (**34**) has the extra commuting flow (**35**).

Remark: Taking into account that $b = z_t = \xi_x$, $c = z_y = \sigma_x$ (see (**22**)) a new potential function Ω can be introduced, where

$$d\Omega = zdx + \xi dt + \sigma dy.$$

Then, the couple of hydrodynamic type system (**34**), (**35**) can be written in the form

$$\Omega_{tt} = u(\Omega_{xx}, \Omega_{xt}, \Omega_{xy}), \qquad \Omega_{yt} = v(\Omega_{xx}, \Omega_{xt}, \Omega_{xy}), \qquad \Omega_{yy} = w(\Omega_{xx}, \Omega_{xt}, \Omega_{xy}).$$

These equations separately can be considered as 2+1 quasilinear equations. However, in general case all of them are non-integrable (see, for instance, [9]). In some cases one of them can be integrable (by the method of hydrodynamic reductions; [9]). Then two other equations are reductions of higher order commuting flows on a three component case. These three 2+1 equations are compatible if (**31**) are fulfilled.

Obviously, the same integrability criterion can be derived for N component hydrodynamic type system written in the conservative form and containing (**5**).

Theorem: *If the Egorov hydrodynamic type system (see* (**1**) *and* (**5**)) *is semi-Hamiltonian, then* $M-2$ *nontrivial commuting flows can be found in quadratures. In such a case these hydrodynamic type systems can be written in the symmetric form* (**22**).

Proof: Consider the Egorov hydrodynamic type system written in the form

$$\partial_t a_1 = \partial_x a_2, \qquad \partial_t a_k = \partial_x b_k(\mathbf{a}), \qquad k = 2, 3, ..., N. \tag{37}$$

If this hydrodynamic type system is semi-Hamiltonian, then $(N-1)\times(N-1)$ symmetric matrix with the elements $a_{kn}(\mathbf{a})$ can be introduced, where $a_{k2}(\mathbf{a}) = b_k(\mathbf{a})$ and all other elements are not determined yet. Since (**37**) is semi-Hamiltonian, then a_1 is the potential of the Egorov metric (see (**9**) and (**21**)), i.e. $\partial_i a_1 = H^2_{(1)i}$. Then (see (**9**)) $\partial_i a_k = H_{(k)i}H_{(1)i}$ and $\partial_i b_k = H_{(k)i}H_{(2)i}$. Then, indeed, all other components a_{kn} can be found in quadratures (see (**20**))

$$da_{kn} = \sum H_{(k)m}H_{(n)m}dr^m = \sum \frac{\partial_m a_k \partial_m a_n}{\partial_m a_1} dr^m.$$

Definition: *The commuting flows* (**15**) *written in the conservative form* (**22**) *are said to be the* ***Egorov basic set***.

Remark: The above proof can be obtained without the Riemann invariants r^k. The compatibility conditions $\partial_{t^k}(\partial_{t^n} a_m) = \partial_{t^n}(\partial_{t^k} a_m)$ lead to the full set of relationships between the coefficients a_{kn}, which are complicated in the variables a^k (see, for instance, the above case (**32**)).

The main reason for consideration of local Hamiltonian structures for the Egorov hydrodynamic type systems (**34**) is following: the general solution (**26**) is determined by the general solution (**33**) (in the three component case), which is parameterized by three arbitrary functions of a single variable. However, in general case this linear PDE system has variable coefficients. The Hamiltonian structure leads to a reduction of (**33**) to the linear ODE system, whose coefficients $h^{(k)}$ can be found recursively (see below).

6 *Orthogonal* curvilinear coordinate nets

Local Hamiltonian structures of hydrodynamic type systems integrable by the generalized hodograph method are connected with the theory of *orthogonal curvilinear coordinate nets* (see [2]). The zero curvature condition (**18**) is a consequence of relationship of two linear problems (**7**) and (**11**)

$$H_i = \partial_i \psi_i + \sum_{m \neq i} \beta_{mi} \psi_m. \tag{38}$$

The existence of this first order transformation is equivalent (see [25]) the existence of local Hamiltonian structure for corresponding hydrodynamic type system (**9**)

$$a_t^i = \partial_x \left(\bar{g}^{ik} \frac{\delta \mathbf{h}}{\delta a^k} \right), \tag{39}$$

where the Hamiltonian is $\mathbf{h} = \int h(\mathbf{a})dx$, $\partial_i h = \psi_i^{(2)} H_{(1)i}$, the momentum density is $P = \bar{g}_{ik} a^i a^k / 2$, $\partial_i P = \psi_i^{(1)} H_{(1)i}$, where

$$H_{(2)i} = \partial_i \psi_i^{(2)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(2)}, \qquad H_{(1)i} = \partial_i \psi_i^{(1)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(1)}$$

and the flat coordinates a^k are determined by its derivatives $\partial_i a^k = \bar{\psi}_i^{(k)} H_{(1)i}$, the **constant** *non-degenerate* metric

$$\bar{g}^{ik} = \Sigma \bar{\psi}_m^{(i)} \bar{\psi}_m^{(k)} \tag{40}$$

is given by

$$0 = \partial_i \bar{\psi}_i^{(k)} + \sum_{m \neq i} \beta_{mi} \bar{\psi}_m^{(k)}, \qquad k = 1, 2, ..., N. \tag{41}$$

If the rotation coefficients β_{ik} are symmetric (see (**17**)) then all above formulas simplify (see [25]). For instance, the zero curvature condition (**18**) reduces to $\delta \beta_{ik} = 0$, where $\delta = \Sigma \partial_m$ is a shift operator. It means, that the rotation coefficients β_{ik} depend only on the differences of the Riemann invariants $r^n - r^m$. The corresponding linear transformation (**38**) reduces to $H_i = \delta \tilde{H}_i$ (where H_i and $\tilde{H}_i$ are solutions of the linear system (**7**)); the linear system (**41**) reduces to $\delta \bar{H}_i^{(k)} = 0$. It means, that the *basic* Lame coefficients $\bar{H}_i^{(k)}$ depend only on the differences of the Riemann invariants $r^n - r^m$. Since the linear problems (**7**) and (**11**) coincide, then we are able to introduce the Lame coefficients with up/sub-indexes

$$\bar{H}_i^{(k)} = \bar{g}^{ks} \bar{H}_{(s)i}, \qquad \bar{H}_{(k)i} = \bar{g}_{ks} \bar{H}_i^{(s)},$$

where the constant *non-degenerate* metric is given by (cf. (**40**))

$$\bar{g}^{ik} = \sum \bar{H}^{(i)m} \bar{H}^{(k)m}, \qquad \bar{g}_{ik} = \sum \bar{H}_{(i)m} \bar{H}_{(k)m}.$$

Let us introduce the *adjoint* flat coordinates $a_i = \bar{g}_{ik} a^k$. Then the Hamiltonian hydrodynamic type systems (**39**) can be written in the form

$$\partial_t a_i = \partial_x \left(\bar{g}_{ik} \frac{\delta \mathbf{h}}{\delta a_k} \right). \tag{42}$$

In this paper we restrict our consideration on two cases, where the the Egorov hydrodynamic type system is written via flat coordinates a^k (see (**39**)). The first case is determined by the condition that a (see (**5**)) is a flat coordinate a_1 and b is a flat coordinate a_2. In general case a is an arbitrary conservation law density h.

7 Local Hamiltonian structures. The complete integrability

Without lost of generality for simplicity we restrict our consideration on three component Egorov hydrodynamic type systems (**39**). Since the symmetric constant matrix $\bar{g}^{ik}$ under a linear transformation of independent variables can be reduced to the diagonal or skew-diagonal case, we restrict our consideration of the first case (a and b are flat coordinates) on these two sub-cases.

1. Let us consider the Egorov hydrodynamic type system (**34**) with the local Hamiltonian structure

$$a_t = \partial_x \frac{\partial h_1}{\partial c}, \quad b_t = \partial_x \frac{\partial h_1}{\partial b}, \quad c_t = \partial_x \frac{\partial h_1}{\partial a}, \tag{43}$$

where $\partial h_1/\partial c = b$ (see (**5**)). Then we are able to choose the extra commuting flow (**35**) written in the same Hamiltonian form

$$a_y = \partial_x \frac{\partial h_2}{\partial c}, \quad b_y = \partial_x \frac{\partial h_2}{\partial b}, \quad c_y = \partial_x \frac{\partial h_2}{\partial a}, \tag{44}$$

where $\partial h_2/\partial c = c$ (see (**29**)).

Definition: *The Egorov basic set of commuting flows* (**22**) *is said to be **canonical** if the first flat coordinate a_1 is a potential of the Egorov metric, and all other flat coordinates a_k are fluxes of the first conservation law $\partial_{t^k} a_1 = \partial_{t^1} a_k$, where the corresponding local Hamiltonian structure is given by* (**42**).

Since $\partial h_2/\partial b = \partial h_1/\partial a$, both Egorov hydrodynamic type systems are determined by the sole function $z(a,b)$ satisfying the famous *associativity equation* (see [3], [6], [8])

$$z_{aaa} = z_{abb}^2 - z_{aab} z_{bbb}, \tag{45}$$

where $h_1 = bc + z_b$, $h_2 = c^2/2 + z_a$ and the momentum density P is given by $h_0 = ac + b^2/2$. Since the flux h of the potential a (see (**5**)) is a conservation law density for an arbitrary commuting flow (see (**6**)), then the shift operator $\partial/\partial c$ is consistent with linear system (**33**) describing conservation law densities (i.e. $\partial\tilde{h}/\partial c = h$, where h and $\tilde{h}$ are solutions of the linear system (**33**)). It is easy to understand, if to take into account that the first equation of an arbitrary commuting flow is written in the form (cf. (**5**) and (**43**))

$$a_\tau = \partial_x \frac{\partial \tilde{h}}{\partial c} = \partial_x h. \tag{46}$$

Theorem: *The Egorov hydrodynamic type system* (**43**) *has three infinite series of conservation law densities, whose coefficients $h_k^{(n)}$ ($k = 0, 1, 2$ and $n = 0, 1, 2, ...$) can be found iteratively.*

Proof: Taking into account (**36**) two other equations of an arbitrary commuting flow are written in the form

$$b_\tau = \partial_x \frac{\partial \tilde{h}}{\partial b} = \partial_x p, \qquad c_\tau = \partial_x \frac{\partial \tilde{h}}{\partial a} = \partial_x q.$$

Thus (see also (**24**)),

$$\begin{aligned} dh_n^{(k+1)} &= h_n^{(k)} dc + p_n^{(k)} db + q_n^{(k)} da, \\ dp_n^{(k+1)} &= p_n^{(k)} dc + \left(q_n^{(k)} + z_{bbb} p_n^{(k)} + z_{abb} h_n^{(k)}\right) db + \left(z_{abb} p_n^{(k)} + z_{aab} h_n^{(k)}\right) da, \\ dq_n^{(k+1)} &= q_n^{(k)} dc + \left(z_{abb} p_n^{(k)} + z_{aab} h_n^{(k)}\right) db + \left(z_{aab} p_n^{(k)} + z_{aaa} h_n^{(k)}\right) da, \end{aligned} \tag{47}$$

where $h_0^{(0)} = a$, $p_0^{(0)} = b$, $q_0^{(0)} = c$; $h_1^{(0)} = b$, $p_1^{(0)} = c + z_{bb}$, $q_1^{(0)} = z_{ab}$; $h_2^{(0)} = c$, $p_2^{(0)} = z_{ab}$, $q_2^{(0)} = z_{aa}$. Then three infinite series of particular solutions by the generalized hodograph method (see (**26**)) are given in an implicit form (cf. (**36**))

$$x = q(a, b, c), \quad t = p(a, b, c), \quad y = h(a, b, c), \tag{48}$$

where

$$h = \sum_{k=0}^{2} \sum_{n=0}^{\infty} \sigma_{kn} h_k^{(n)}, \qquad p = \sum_{k=0}^{2} \sum_{n=0}^{\infty} \sigma_{kn} p_k^{(n)}, \qquad q = \sum_{k=0}^{2} \sum_{n=0}^{\infty} \sigma_{kn} q_k^{(n)}$$

and σ_{kn} are arbitrary constants.

Complete integrability: In general N component case the Egorov basic set (**22**) is canonical (see (**39**) and (**42**))

$$\partial_{t^k} a_n = \partial_{t^1} \left(\frac{\partial h_k}{\partial a^n} \right) = \partial_{t^1} a_{kn}(\mathbf{a}). \tag{49}$$

It means, that

$$h_k = \frac{\partial F}{\partial a^k}.$$

The compatibility conditions $\partial_{t^k}(\partial_{t^m} a_n) = \partial_{t^m}(\partial_{t^k} a_n)$ lead to the WDVV equation (see [3])

$$\frac{\partial^3 F}{\partial a^k \partial a^i \partial a^s} \bar{g}^{sp} \frac{\partial^3 F}{\partial a^p \partial a^j \partial a^n} = \frac{\partial^3 F}{\partial a^j \partial a^i \partial a^s} \bar{g}^{sp} \frac{\partial^3 F}{\partial a^p \partial a^k \partial a^n}.$$

The canonical Egorov basic set must have N infinite series of conservation laws

$$\partial_{t^k} h_s^{(p)} = \partial_{t^1} q_{k,s}^{(p)}, \quad k, s = 1, 2, ..., N, \quad p = 0, 1, 2, ... \tag{50}$$

and commuting flows (see [25])

$$\partial_{t_p^s} a_k = \partial_{t^1} \left(\frac{\partial h_s^{(p)}}{\partial a^k} \right), \quad k, s = 1, 2, ..., N, \quad p = 0, 1, 2, ... \tag{51}$$

Since a_1 is a potential of the Egorov metric, then (see (6))

$$\partial_{t_p^s} a_1 = \partial_{t^1} \left(\frac{\partial h_s^{(p)}}{\partial a^1} \right) = \partial_{t^1} h_s^{(p-1)}, \tag{52}$$

where $h_s^{(0)} \equiv a_s$ and $t_{(1)}^s \equiv t^s$, $s = 1, 2, ..., N$. Thus, $\partial h_s^{(p)} / \partial a^1 = h_s^{(p)}$. The compatibility conditions $\partial_{t^k} (\partial_{t_p^s} a_1) = \partial_{t_p^s} (\partial_{t^k} a_1)$ lead to (see (50) and (51))

$$\partial_{t^k} h_s^{(p-1)} = \partial_{t^1} \frac{\partial h_s^{(p)}}{\partial a^k} = \partial_{t^1} q_{k,s}^{(p-1)}. \tag{53}$$

Thus, the conservation law densities $h_s^{(p+1)}$ can be found in quadratures (see (52) and (53))

$$dh_s^{(p+1)} = q_{k,s}^{(p)} da^k, \qquad s, k = 1, 2, ..., N, \;\; p = 0, 1, 2, ..., \tag{54}$$

where $q_{1,s}^{(p)} \equiv h_s^{(p)}$. The consistency of the conservation laws (53) with (49) lead to the relationships

$$\frac{\partial q_{k,s}^{(p)}}{\partial a^i} = \frac{\partial h_s^{(p)}}{\partial a^n} \bar{g}^{nm} \frac{\partial^3 F}{\partial a^m \partial a^k \partial a^i}. \tag{55}$$

The substitution of (54) in r.h.s. of (55) implies the recursion relationship

$$\frac{\partial^2 h_{k,s}^{(p+1)}}{\partial a^i \partial a^k} = \frac{\partial^3 F}{\partial a^i \partial a^k \partial a^m} \bar{g}^{mn} \frac{\partial h_s^{(p)}}{\partial a^n}, \qquad n = 0, 1, 2, ...,$$

found by B.A. Dubrovin in [3]. Taking into account (54) the above formula leads to the iterative procedure (cf. (47))

$$dq_{k,s}^{(p+1)} = q_{n,s}^{(p)} \bar{g}^{nm} \frac{\partial^3 F}{\partial a^m \partial a^k \partial a^i} da^i, \qquad i, k, s, m, n = 1, 2, ..., N, \;\; p = 0, 1, 2, ...,$$

including (54). The complete integrability of the Egorov Hamiltonian hydrodynamic type systems (49) was proved in [25]. Then the general solution can be given in implicit form by the generalized hodograph method (see (26) and cf. (48))

$$t^k = \bar{g}^{kn} \frac{\partial h}{\partial a^n}, \qquad h = \sum_{s=0}^{N} \sum_{p=0}^{\infty} \sigma_{ps} h_s^{(p)},$$

where σ_{ps} are appropriate constants.

Example: The first choice $h_0^{(0)} = a$, $p_0^{(0)} = b$, $q_0^{(0)} = c$ determines the Egorov hydrodynamic type system

$$\partial_{\tau_0^{(0)}} a = \partial_x \left(ac + \frac{b^2}{2} \right), \quad \partial_{\tau_0^{(0)}} b = \partial_x \left(bc + bz_{bb} + az_{ab} - z_b \right), \quad \partial_{\tau_0^{(0)}} c = \partial_x \left(\frac{c^2}{2} + bz_{ab} + az_{aa} - z_a \right).$$

Thus, first four conservation laws for the canonical Egorov basic set (43), (44) together with the above commuting flow can be written in the potential symmetric form

$$d \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{pmatrix} = \begin{pmatrix} a & b & c & P \\ b & u & v & R \\ c & v & w & S \\ P & R & S & Q \end{pmatrix} d \begin{pmatrix} x \\ t \\ y \\ \tau \end{pmatrix}, \tag{56}$$

where all coefficients can be found in quadratures (see (47))

$$\begin{aligned} u &= c + z_{bb}, \qquad v = z_{ab}, \qquad w = z_{aa}, \\ R &= bc + bz_{bb} + az_{ab} - z_b, \qquad S = \frac{c^2}{2} + bz_{ab} + az_{aa} - z_a, \\ Q &= ac^2 + b^2c + a^2z_{aa} + 2abz_{ab} + b^2z_{bb} - 2(az_a + bz_b - z). \end{aligned} \tag{57}$$

2. Let us consider the Egorov hydrodynamic type system (34) with the local Hamiltonian structure

$$a_t = \partial_x \frac{\partial h_1}{\partial a}, \quad b_t = \partial_x \frac{\partial h_1}{\partial c}, \quad c_t = \partial_x \frac{\partial h_1}{\partial b}, \tag{58}$$

where $\partial h_1/\partial a = b$ (see (5)). Then we are able to choose the extra commuting flow (35) written in the same Hamiltonian form

$$a_y = \partial_x \frac{\partial h_2}{\partial a}, \quad b_y = \partial_x \frac{\partial h_2}{\partial c}, \quad c_y = \partial_x \frac{\partial h_2}{\partial b},$$

where $\partial h_2/\partial a = c$ (see (29)). Since $\partial h_2/\partial c = \partial h_1/\partial b$, both Egorov hydrodynamic type systems are determined by the sole function $z(b, c)$ satisfying the famous *associativity equation* (see [3], [6], [8])

$$1 + z_{bbc}z_{bcc} = z_{bbb}z_{ccc},$$

where $h_1 = ab + z_c$, $h_2 = ac + z_b$ and the momentum density P is given by $h_0 = a^2/2 + bc$.

Theorem: *The hydrodynamic type system (43) is equivalent to the hydrodynamic type system (58) under the transformation $x \leftrightarrow t$.*

Proof: Let us replace $x \leftrightarrow t$, $\partial h_1/\partial c \to \tilde{a}$, $\partial h_1/\partial b \to \tilde{c}$, $\partial h_1/\partial a \to \tilde{b}$; $c \to \partial\tilde{h}_1/\partial\tilde{a}$, $a \to \partial\tilde{h}_1/\partial\tilde{b}$, $b \to \partial\tilde{h}_1/\partial\tilde{c}$. Then (58) transforms in (43).

Remark: The transformation connecting the above associativity equations was found in [8]).

3. In general case the Egorov hydrodynamic type system (39) has the couple of extra conservation laws (5), where a and b are not connected with the Hamiltonian structure (39).

Let us consider the commuting flow (see (39) and (42))

$$\partial_z a_i = \partial_x(\partial\tilde{h}/\partial a^i),$$

where the extra conservation law is (see (5))

$$\partial_z a = \partial_x a_1. \tag{59}$$

Under the transformation (see [21]) $x \leftrightarrow z$, the above commuting flow reduces to the Egorov hydrodynamic type system

$$\partial_x c_i = \partial_z(\partial\bar{h}/\partial c^i), \tag{60}$$

where $c_i = \partial\tilde{h}/\partial a^k$, $\bar{h} = a^k\partial\tilde{h}/\partial a^k - \tilde{h}$ and the extra conservation law is (cf. (59))

$$\partial_x(\partial\bar{h}/\partial c^1) = \partial_z a,$$

where $a_1 = \partial\bar{h}/\partial c^1$. Thus, in this case the potential of the Egorov metric $\bar{a} \equiv c_1$. At the same time the Egorov hydrodynamic type system (60) possesses the commuting flow

$$\partial_y c_i = \partial_z(\partial \underline{\mathrm{h}}/\partial c^i),$$

where the extra conservation law is

$$\partial_y c_1 = \partial_z c_2.$$

Thus, this general case is reduced to the simplest case described above.

8 Reciprocal Transformations

In the previous section we considered the simplest reciprocal transformation

$$dy^k = \sigma_n^k dt^n, \qquad \sigma_n^k = \mathrm{const}$$

preserving the Egorov hydrodynamic type systems. In this section a more complicated reciprocal transformation is presented. Suppose we already know (see, for instance, the previous section) all conservation laws and commuting flows for the Egorov hydrodynamic type systems (22). The generalized reciprocal transformation (see [11]) contains M rows and M columns determined by M conservation laws and $M-1$ commuting flows. The number M and the number N (see (1)) do not correlate to each other in general case. The Egorov hydrodynamic type system (22) can be written in the potential symmetric form (19). If $M \geqslant N$, then the generalized symmetric reciprocal transformation (cf. (19))

$$d\tilde{y}_i = a_{ik}(\mathbf{a})dt^k$$

preserves a potential symmetric form and again transforms (22) to the Egorov hydrodynamic type systems. If $M < N$, then the generalized reciprocal transformation

$$d\tilde{y}_i = a_{ik}(\mathbf{a})dt^k, \qquad i = 1, 2, ..., M; \qquad d\tilde{y}_i = \sigma_{ik} dt^k, \qquad i = M+1, ..., N$$

must contain a symmetric part including the potential a of the Egorov metric, while all other independent variables transform linearly (i.e. $\sigma_{ik} = \mathrm{const}$).

Without lost of generality we restrict our consideration for simplicity on the reciprocal transformation

$$dy^1 = a_k dt^k, \qquad dy^k = dt^k, \qquad k = 2, 3, ..., N. \tag{61}$$

Then the Egorov hydrodynamic type system (22) reduces to the similar set of the Egorov hydrodynamic type systems

$$\partial_{y^k}\left(-\frac{1}{a_1}\right) = \partial_{y^1}\frac{a_k}{a_1}, \qquad \partial_{y^k}\frac{a_n}{a_1} = \partial_{y^1}\left(a_{nk} - \frac{a_n a_k}{a_1}\right), \qquad k, n = 2, 3, ..., N.$$

A recalculation of local Hamiltonian structures under generalized reciprocal transformation is given in [11]. In this section we restrict our consideration on the above three component case (43).

Theorem: *The local Hamiltonian structure* (**43**) *of the Egorov hydrodynamic type system* (**34**) *is invariant under the reciprocal transformation* (**61**)

$$dz = adx + bdt, \qquad dy = dt. \tag{62}$$

Proof: The Egorov hydrodynamic type system (**34**) has 4 local conservation laws associated with local Hamiltonian structure (**43**), where the momentum density P is the quadratic expression with respect to flat coordinates (see [23])

$$P = ac + \frac{b^2}{2}. \tag{63}$$

Under the above reciprocal transformation *any* conservation law density h reduces to h/a^1. Thus, (**63**) reduces to

$$\tilde{P} = \tilde{a}\tilde{c} + \frac{\tilde{b}^2}{2},$$

where $\tilde{P} = -c/a$, $\tilde{a} = 1/a$, $\tilde{c} = -P/a$, $\tilde{b} = -b/a$.

Remark: The Hamiltonian density of the Egorov hydrodynamic type system (**43**)

$$\tilde{a}_t = \partial_z \frac{\partial \tilde{h}}{\partial \tilde{c}}, \qquad \tilde{b}_t = \partial_z \frac{\partial \tilde{h}}{\partial \tilde{b}}, \qquad \tilde{c}_t = \partial_z \frac{\partial \tilde{h}}{\partial \tilde{a}}$$

is given by $\tilde{h} = h/a$.

9 Three orthogonal Egorov curvilinear coordinate systems

N orthogonal Egorov curvilinear coordinate nets are described by the Bianchi–Darboux–Egorov–Lame system (**8**)

$$\partial_i \beta_{jk} = \beta_{ji}\beta_{ik}, \quad i \neq j \neq k; \qquad \beta_{ik} = \beta_{ki}, \quad i \neq k, \qquad \delta\beta_{ik} = 0.$$

N orthogonal curvilinear coordinate nets were investigated in many publications (see, for instance, [1], [2], [12], [25], [27]; and plenty references therein). In this section we establish a new link connecting an infinite set of N orthogonal Egorov curvilinear coordinate systems:

$$... \leftarrow \beta_{ik}^{(-2)} \leftarrow \beta_{ik}^{(-1)} \leftarrow \beta_{ik}^{(0)} \rightarrow \beta_{ik}^{(1)} \rightarrow \beta_{ik}^{(2)} \rightarrow ... \tag{64}$$

Without lost of generality we restrict our consideration for simplicity on a three component case only. The main advantage of the approach presented below is that all formulas are given via **flat** coordinates only (in comparison with the Riemann invariants). In such a case corresponding formulas can be easily used for a construction of solutions for the WDVV equation.

Suppose all rotations coefficients $\beta_{ik}^{(0)}$ and the Lame coefficients $H_{(k)i}$ are given. It means that the couple of the Egorov hydrodynamic type systems (**29**) is given too. The local Hamiltonian structure (**39**) can be reduced (in three component case) to (**43**) by a linear transformation of field variables ($a^k \to \sigma^k_s a^s$, $\sigma^k_s =$ const), because the symmetric constant *non-degenerate* metric $\bar{g}^{ik}$ is reducible to the diagonal or skew-diagonal form.

1. Let us apply the reciprocal transformation (**61**)

$$\begin{aligned} dx^{(1)} &= a^{(0)}dx^{(0)} + b^{(0)}dt^{(0)} + c^{(0)}dy^{(0)} + \left(a^{(0)}c^{(0)} + \frac{\left(b^{(0)}\right)^2}{2}\right)d\tau^{(0)}, \\ dt^{(1)} &= dt^{(0)}, \qquad d\tau^{(1)} = dy^{(0)}, \qquad dy^{(1)} = d\tau^{(0)} \end{aligned} \tag{65}$$

to the Egorov hydrodynamic type systems (**29**) (see also (**43**), (**44**)) written for simplicity in the potential symmetric form (**56**)

$$d\begin{pmatrix} x^{(1)} \\ \tilde{t}^{(1)} \\ \tilde{y}^{(1)} \\ \tilde{\tau}^{(1)} \end{pmatrix} = \begin{pmatrix} a^{(0)} & b^{(0)} & c^{(0)} & P^{(0)} \\ b^{(0)} & u^{(0)} & v^{(0)} & R^{(0)} \\ c^{(0)} & v^{(0)} & w^{(0)} & S^{(0)} \\ P^{(0)} & R^{(0)} & S^{(0)} & Q^{(0)} \end{pmatrix} d\begin{pmatrix} x^{(0)} \\ t^{(0)} \\ y^{(0)} \\ \tau^{(0)} \end{pmatrix},$$

where $z^{(0)}$ satisfies the associativity equation (**45**). The hydrodynamic type systems (**29**) form the canonical Egorov basic set (**43**), (**44**) (i.e. the fluxes $b^{(0)}$, $c^{(0)}$ of the first conservation law $\partial_{t^{(0)}}a^{(0)} = \partial_{x^{(0)}}b^{(0)}$, $\partial_{y^{(0)}}a^{(0)} = \partial_{x^{(0)}}c^{(0)}$ are the corresponding flat coordinates). Since $a^{(1)} = 1/a^{(0)}$, $b^{(1)} = -b^{(0)}/a^{(0)}$, $c^{(1)} = -P^{(0)}/a^{(0)}$ are flat coordinates (and $a^{(1)}$ is a potential of the *transformed* Egorov metric), then the *transformed* canonical Egorov basic set (**22**) can be extracted from the above *transformed* potential symmetric form

$$d\begin{pmatrix} x^{(0)} \\ -\tilde{t}^{(1)} \\ -\tilde{\tau}^{(1)} \\ -\tilde{y}^{(1)} \end{pmatrix} = \begin{pmatrix} 1/a^{(0)} & -\frac{b^{(0)}}{a^{(0)}} & -\frac{P^{(0)}}{a^{(0)}} & -\frac{c^{(0)}}{a^{(0)}} \\ -\frac{b^{(0)}}{a^{(0)}} & \frac{\left(b^{(0)}\right)^2}{a^{(0)}} - u^{(0)} & \frac{b^{(0)}P^{(0)}}{a^{(0)}} - R^{(0)} & \frac{b^{(0)}c^{(0)}}{a^{(0)}} - v^{(0)} \\ -\frac{P^{(0)}}{a^{(0)}} & \frac{b^{(0)}P^{(0)}}{a^{(0)}} - R^{(0)} & \frac{\left(P^{(0)}\right)^2}{a^{(0)}} - Q^{(0)} & \frac{c^{(0)}P^{(0)}}{a^{(0)}} - S^{(0)} \\ -\frac{c^{(0)}}{a^{(0)}} & \frac{b^{(0)}c^{(0)}}{a^{(0)}} - v^{(0)} & \frac{c^{(0)}P^{(0)}}{a^{(0)}} - S^{(0)} & \frac{\left(c^{(0)}\right)^2}{a^{(0)}} - w^{(0)} \end{pmatrix} d\begin{pmatrix} x^{(1)} \\ t^{(1)} \\ \tau^{(1)} \\ y^{(1)} \end{pmatrix}.$$

The transformed canonical Egorov basic set must be given by (see (**43**), (**44**))

$$\begin{aligned} &\partial_{t^{(1)}}a^{(1)} = \partial_{x^{(1)}}b^{(1)}, && \partial_{y^{(1)}}a^{(1)} = \partial_{x^{(1)}}c^{(1)}, \\ &\partial_{t^{(1)}}b^{(1)} = \partial_{x^{(1)}}\left(c^{(1)} + z^{(1)}_{b^{(1)}b^{(1)}}\right), && \partial_{y^{(1)}}b^{(1)} = \partial_{x^{(1)}}z^{(1)}_{a^{(1)}b^{(1)}}, \\ &\partial_{t^{(1)}}c^{(1)} = \partial_{x^{(1)}}z^{(1)}_{a^{(1)}b^{(1)}}, && \partial_{y^{(1)}}c^{(1)} = \partial_{x^{(1)}}z^{(1)}_{a^{(1)}a^{(1)}}, \end{aligned} \tag{66}$$

where $z^{(1)}$ satisfies the associativity equation (**45**). Comparing corresponding fluxes from the above equations

$$z^{(1)}_{a^{(1)}a^{(1)}} = \frac{(P^{(0)})^2}{a^{(0)}} - Q^{(0)}, \qquad z^{(1)}_{a^{(1)}b^{(1)}} = \frac{b^{(0)}P^{(0)}}{a^{(0)}} - R^{(0)}, \qquad c^{(1)} + z^{(1)}_{b^{(1)}b^{(1)}} = \frac{(b^{(0)})^2}{a^{(0)}} - u^{(0)},$$

one can compute the new solution $z^{(1)}$ of the associativity equation (45) in quadratures

$$dz^{(1)} = z^{(1)}_{a^{(1)}} d\frac{1}{a^{(0)}} - z^{(1)}_{b^{(1)}} d\frac{b^{(0)}}{a^{(0)}},$$

where

$$dz^{(1)}_{a^{(1)}} = \left(\frac{\left(P^{(0)}\right)^2}{a^{(0)}} - Q^{(0)}\right) d\frac{1}{a^{(0)}} - \left(\frac{b^{(0)}P^{(0)}}{a^{(0)}} - R^{(0)}\right) d\frac{b^{(0)}}{a^{(0)}},$$

$$dz^{(1)}_{b^{(1)}} = \left(\frac{b^{(0)}P^{(0)}}{a^{(0)}} - R^{(0)}\right) d\frac{1}{a^{(0)}} - \left(\frac{(b^{(0)})^2}{a^{(0)}} - u^{(0)} + \frac{P^{(0)}}{a^{(0)}}\right) d\frac{b^{(0)}}{a^{(0)}}.$$

Moreover, the substitution (57) in the above differentials yields an explicit link

$$z^{(1)} = \frac{(b^{(0)})^4}{8(a^{(0)})^3} - \frac{1}{(a^{(0)})^2} z^{(0)}. \qquad (67)$$

of the solutions $z^{(0)}$ and $z^{(1)}$ of the associativity equation (45).

Remark: *The Ribaucour transformation.* Transformations of local Hamiltonian structures under generalized reciprocal transformations are described in [11]. In the simplest case (65) the comparison of the Lame coefficients $\bar{H}^{(0)}_{(1)i}$ and the *transformed* Lame coefficients $\bar{H}^{(1)}_{(1)i}$ from (see (20))

$$\partial_i a^{(0)} = (\bar{H}^{(0)}_{(1)i})^2, \qquad \partial_i a^{(1)} \equiv \partial_i \frac{1}{a^{(0)}} = -\frac{1}{(a^{(0)})^2} (\bar{H}^{(0)}_{(1)i})^2 = (\bar{H}^{(1)}_{(1)i})^2$$

leads to the Ribaucour transformation (see details in [25]) of the three orthogonal curvilinear coordinate nets

$$\beta^{(1)}_{ik} = \beta^{(0)}_{ik} - \frac{\bar{H}^{(0)}_{(1)i}\bar{H}^{(0)}_{(1)k}}{a^{(0)}}, \qquad \bar{H}^{(1)}_{(1)k} = i\frac{\bar{H}^{(0)}_{(1)k}}{a^{(0)}}.$$

These rotation coefficients $\beta^{(0)}_{ik}$ and $\beta^{(0)}_{ik}$ are symmetric and satisfy the zero curvature condition (18). Thus, indeed, the Egorov three orthogonal curvilinear coordinate nets are preserved under the above Ribaucour transformation (see [1] and [25]).

2. Let us rewrite the canonical Egorov basic set (66) in the form

$$\partial_{t^{(1)}} z^{(1)}_{a^{(1)}a^{(1)}} = \partial_{y^{(1)}} z^{(1)}_{a^{(1)}b^{(1)}}, \qquad \partial_{x^{(1)}} z^{(1)}_{a^{(1)}a^{(1)}} = \partial_{y^{(1)}} c^{(1)},$$

$$\partial_{t^{(1)}} z^{(1)}_{a^{(1)}b^{(1)}} = \partial_{y^{(1)}} \left(c^{(1)} + z^{(1)}_{b^{(1)}b^{(1)}}\right), \qquad \partial_{x^{(1)}} z^{(1)}_{a^{(1)}b^{(1)}} = \partial_{y^{(1)}} b^{(1)},$$

$$\partial_{t^{(1)}} c^{(1)} = \partial_{y^{(1)}} b^{(1)}, \qquad \partial_{x^{(1)}} c^{(1)} = \partial_{y^{(1)}} a^{(1)}.$$

Thus, under the linear transformation of independent variables

$$x^{(2)} = y^{(1)}, \qquad t^{(2)} = t^{(1)}, \qquad y^{(2)} = x^{(1)} \qquad (68)$$

the above canonical Egorov basic set reduces to the *transformed* canonical Egorov basic set

$$\partial_{t^{(2)}}a^{(2)} = \partial_{x^{(2)}}b^{(2)}, \qquad \partial_{y^{(2)}}a^{(2)} = \partial_{x^{(2)}}c^{(2)},$$

$$\partial_{t^{(2)}}b^{(2)} = \partial_{x^{(2)}}\left(c^{(2)} + z^{(2)}_{b^{(2)}b^{(2)}}\right), \qquad \partial_{y^{(2)}}b^{(2)} = \partial_{x^{(2)}}z^{(2)}_{a^{(2)}b^{(2)}},$$

$$\partial_{t^{(2)}}c^{(2)} = \partial_{x^{(2)}}z^{(2)}_{a^{(2)}b^{(2)}}, \qquad \partial_{y^{(2)}}c^{(2)} = \partial_{x^{(2)}}z^{(2)}_{a^{(2)}a^{(2)}},$$

where

$$a^{(2)} = z^{(1)}_{a^{(1)}a^{(1)}}, \quad b^{(2)} = z^{(1)}_{a^{(1)}b^{(1)}}, \quad c^{(2)} = c^{(1)}, \quad z^{(2)}_{b^{(2)}b^{(2)}} = z^{(1)}_{b^{(1)}b^{(1)}}, \quad z^{(2)}_{a^{(2)}b^{(2)}} = b^{(1)}, \quad z^{(2)}_{a^{(2)}a^{(2)}} = a^{(1)}.$$

It means that the solution $z^{(2)}$ of the associativity equation (**45**) can be found in quadratures

$$dz^{(2)} = z^{(2)}_{a^{(2)}}dz^{(1)}_{a^{(1)}a^{(1)}} + z^{(2)}_{b^{(2)}}dz^{(1)}_{a^{(1)}b^{(1)}}, \tag{69}$$

where

$$z^{(2)}_{a^{(2)}} = a^{(1)}z^{(1)}_{a^{(1)}a^{(1)}} + b^{(1)}z^{(1)}_{a^{(1)}b^{(1)}} - z^{(1)}_{a^{(1)}}, \qquad dz^{(2)}_{b^{(2)}} = z^{(1)}_{b^{(1)}b^{(1)}}dz^{(1)}_{a^{(1)}b^{(1)}} + b^{(1)}dz^{(1)}_{a^{(1)}a^{(1)}}.$$

Remark: The above right differential

$$dG = z_{bb}dz_{ab} + bdz_{aa}$$

exists iff the function z is a solution of the associativity equation (**45**). This function G is the first such example of infinitely many expressions, which cannot be computed explicitly. A computation of higher conservation laws leads to similar differentials. For instance (see (**43**), (**44**)),

$$\partial_y(bc + z_b) = \partial_x(cz_{ab} + G), \qquad \partial_t\left(\frac{c^2}{2} + z_a\right) = \partial_x[z_{ab}(c + z_{bb}) + bz_{aa} - G].$$

Thus, the *iterative replication* of solutions for the associativity equation (**45**) can be based on the above two steps

$$... \leftarrow z^{(-2)} \leftarrow z^{(-1)} \leftarrow z^{(0)} \rightarrow z^{(1)} \rightarrow z^{(2)} \rightarrow ... \tag{70}$$

The *negative* direction means that the transformations (**65**) and (**68**) are applied in an inverse order.

The **main statement** of this section: *Each solution of the associativity equation* (**45**) *creates the canonical Egorov basic set, simultaneously, the corresponding basic Lame coefficients* $\bar{H}_i^{(k)}$ *and rotation coefficients* β_{ik}. *It means, that an infinite set of solutions* $z^{(k)}$ (*see* (**67**), (**69**) *and* (**70**)) *creates an infinite set of orthogonal curvilinear coordinate nets* (**64**). *Thus, a description of all solutions of the associativity equation* (**45**) *is equivalent to a description of three orthogonal curvilinear coordinate nets.*

This set of transformations has the same interpretation as the so-called "dressing method" in the soliton theory.

Choosing the initial solution of the associativity equation (45) $z^{(0)} = 0$, the first iteration yields the solution given by

$$z^{(1)} = \frac{b^4}{8a}.$$

The second iteration yields the same solution. The fourth iteration yields again the "zero" solution. However, in general case this iterative chain of transformations cannot be truncated.

In this paper just a three component case was considered in details, but without any restrictions this approach can be applied for N component case.

Remark: The first transformation (67) is exactly the "inversion I" transformation for the WDVV equation found in [3] (see the formulas B.13, Appendix **B**). The second transformation is exactly the Legendre type transformation for the WDVV equation found in [3] (see the formulas B.1 and B.2, Appendix **B**).

10 Nonlocal Hamiltonian structures

Theory of nonlocal Hamiltonian structures for hydrodynamic type systems was constructed by E.V. Ferapontov in [7] (see also [16]). The simplest nonlocal Hamiltonian structure is associated with the metric of constant curvature (see also [10], [22]). The existence of the Hamiltonian structure for the Egorov hydrodynamic type systems (22) leads to the symmetry operator (see (46) for the local Hamiltonian structure). In this section without lost of generality we restrict our consideration for simplicity on the nonlocal Hamiltonian structure associated with the metric of constant curvature. The corresponding hydrodynamic type system can be written via special field variables a^k in the conservative form (cf. (39))

$$a_t^i = \partial_x \left[(\bar{g}^{ik} - \varepsilon a^i a^k) \frac{\partial h}{\partial a^k} + \varepsilon a^i h \right],$$

where $\bar{g}^{ik}$ is a constant symmetric matrix and ε is a constant curvature.

Let us apply the reciprocal transformation (62) to the Egorov hydrodynamic type system (58). Then the quadratic relationship (connected with local Hamiltonian structure, see [23])

$$P = bc + \frac{a^2}{2}$$

transforms in another type quadratic relationship (connected with the above nonlocal Hamiltonian structure; see [16] and [22])

$$\frac{1}{2} = a_1 a_2 - a_3 a_4,$$

where $a_1 = 1/a$, $a_2 = P/a$, $a_3 = -b/a$, $a_4 = -c/a$. The coordinate a_1 is the potential of the Egorov metric, and the corresponding transformed Egorov hydrodynamic type system has the above nonlocal Hamiltonian structure. Thus, this nonlocal Hamiltonian structure is reducible to the local Hamiltonian structure (39).

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