

Newton polygons for finding exact solutions

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Abstract

A method for finding exact solutions of nonlinear differential equations is presented. Our method is based on the application of the Newton polygons corresponding to nonlinear differential equations. It allows one to express exact solutions of the equation studied through solutions of another equation using properties of the basic equation itself. The ideas of power geometry are used and developed. Our approach has a pictorial rendition, which is illustrative and effective. The method can be also applied for finding transformations between solutions of the differential equations. To demonstrate the method application exact solutions of several equations are found. These equations are: the Korteweg – de Vries – Burgers equation, the generalized Kuramoto - Sivashinsky equation, the fourth – order nonlinear evolution equation, the fifth – order Korteweg – de Vries equation, the modified Korteweg – de Vries equation of the fifth order and nonlinear evolution equation of the sixth order for the turbulence description. Some new exact solutions of nonlinear evolution equations are given.

Keywords: power geometry, exact solution, nonlinear differential equation, traveling wave.

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1 Introduction

One of the most important problems of nonlinear models analysis is the construction of their partial solutions. Nowadays this problem is intensively studied. We know that the inverse scattering transform [1–3] and the Hirota method [3–5] are very useful in looking for the solutions of exactly solvable nonlinear equations. However most of nonlinear equations are nonintegrable ones. For constructing solutions of such equations the following methods are used: the singular manifold method [7–13], the Weierstrass function method [13–15], the tanh–function method [16–21], the Jacobian elliptic function method [22–24] and the trigonometric function method [25,26]. Most of these methods proceed from an a priori

expression for an unknown solution. Usually this expression is a polynomial in elementary or special functions (for example, in trigonometric, Weierstrass, Jacobian and some other). As a result all solutions outside the given family are lost. This disadvantage significantly decreases the strength of such methods.

Lately it was made an attempt to generalize most of these methods and as a result the simplest equation method appeared [27, 28]. Two ideas lay in the basis of this method. The first one was to use an equation of lesser order with known general solution for finding exact solutions. The second one was to take into account possible movable singularities of the original equation. Virtually both ideas existed though not evidently in some methods suggested earlier.

However the method introduced in [27, 28] has one essential disadvantage concerning with an indeterminacy of the simplest equation choice. This disadvantage considerably decreases the effectiveness of the method. In this paper we present a new approach for finding exact solutions of nonlinear differential equations, which greatly expands the method [27, 28] and which is free from disadvantage mentioned above. The method does not postulate the simplest equation but it allows one to find its structure using the properties of the equation studied itself. Our method develops the ideas of power geometry [29–32]. With a help of the power geometry we show that the search of the simplest equation becomes illustrative and effective. Also it is important to mention that the results obtained are sufficiently general and can be applied not only to finding exact solutions but also to constructing transformations for nonlinear differential equations.

The outline of this paper is as follows. The application of the Newton polygons of nonlinear differential equations and the method algorithm for finding exact solutions is presented in section 2. Our method is applied to look for exact solutions of the Korteweg – de Vries – Burgers equation, nonlinear evolution equation of the fifth order and the generalized Kuramoto – Sivashinsky equations in sections 3, 4 and 5 accordingly. Self similar solutions of the fifth order Korteweg – de Vries equations are considered in sections 6 and 7. Solitary and periodical waves of sixth order nonlinear evolution equation for description of the turbulent processes are found in sections 8 and 9.

2 Method applied

Let us assume that we look for exact solutions of the following nonlinear n – order ODE

$$M_n[y(z), y_z(z), y_{zz}(z), \dots, z] = 0. \quad (2.1)$$

In power geometry any differential equation is regarded as a sum of ordinary and differential monomials. Every monomial can be associated with a point on the plane according to the following rules

$$C_1 z^{q_1} y^{q_2} \longrightarrow (q_1, q_2), \quad C_2 \frac{d^k y}{dz^k} \longrightarrow (-k, 1). \quad (2.2)$$

Here C_1, C_2 are arbitrary constants. When monomials are multiplied their coordinates are added. The set of points corresponding to all monomials of a differential equation form its carrier. Having connected the points of the carrier into the convex figure we obtain a convex polygon called the Newton polygon

of differential equation. Thus nonlinear ODE (2.1) can be characterized by the Newton polygon L_1 on the plane. The periphery of the polygon consists of vertexes and edges. By $\{\Gamma_j^{(0)}\}$ let us denote the vertexes and by $\{\Gamma_j^{(1)}\}$ the edges. Most of edges and vertexes of the Newton polygon define power or non – power asymptotics and power expansions for the solutions of the equation [29–32].

Now let us assume that a solution $y(z)$ of the studied equation can be expressed through solutions $Y(z)$ of another equation. The latter equation is called the simplest equation. Consequently we have a relation between $y(z)$ and $Y(z)$

$$y(z) = F(Y(z), Y_z(z), \dots, z). \quad (2.3)$$

The main problem is to find the simplest equation. Substitution (2.3) into basic equation (2.1) yields a transformed differential equation, which is in its turn characterized by the polygon L_2 . By L_3 denote the Newton polygon corresponding to the simplest equation. Analyzing the Newton polygon L_2 we should construct the polygon L_3 . Let the edge $\Gamma_1^{(1)}$ belong to L_2 and the edge $\tilde{\Gamma}_1^{(1)}$ belong to L_3 . It can be proved that if $\Gamma_1^{(1)}$ and $\tilde{\Gamma}_1^{(1)}$ have equal external normal vectors, then corresponding asymptotics coincide accurate to the numerical parameter. Under the term external vector we mean the vector that goes out of the Newton polygon. Consequently, suitable Newton polygon L_3 has all or certain part of edges parallel to those of L_2 . Besides that when the Newton polygon L_3 moves along the plane his apexes should cover the support of the transformed differential equation. From this fact we see in particular that suitable L_3 is of equal or lesser area than L_2 . Let us suppose that we have found such polygon L_3 . Then we can write out the simplest equation

$$E_m(Y(z), Y_z(z), \dots, z) = 0. \quad (2.4)$$

It is important to mention that the choice of the simplest equation is not unique. If the following correlation

$$M_n(F(Y, Y_z, \dots, z)) = \hat{R} E_m(Y, Y_z, \dots, z) \quad (2.5)$$

(where \hat{R} is a differential operator) is true then it means that for any solution $Y(z)$ of the simplest equation (2.4) there exists a solution of (2.1). Generally speaking, any nonlinear solvable differential equation can be the simplest equation. The only requirement is the following: the order of the simplest equation should be lesser than the order of the transformed differential equation. But the most important simplest equations are those that have solutions without movable critical points. If the general solution (or a partial solution) of the simplest equation can be found, then we get explicit representation for a solution of the equation studied. Otherwise we have only the relation (2.3) between solutions of (2.1) and (2.4).

The most useful examples of the simplest equations are the following: the Riccati equation

$$Y_z + Y^2 - a(z)Y - b(z) = 0, \quad (2.6)$$

the equation for the Jacobi elliptic functions

$$R_z^2 = -4R^4 + aR^3 + bR^2 + cR + d, \quad (2.7)$$

and the equation for the Weiershrass elliptic functions

$$R_z^2 = -2R^3 + aR^2 + bR + c. \quad (2.8)$$

Solutions of equations (2.6), (2.7), and (2.8) do not have movable critical points.

Now we are able to state our method. It is composed of six steps.

The first step. Construction of the Newton polygon L_1 , which corresponds to the equation studied.

The second step. Determination of the movable pole order for solutions of the equation studied and transformation of this equation using the expression (2.3).

The third step. Construction of the Newton polygon L_2 corresponding to the transformed equation.

The fourth step. Construction of the Newton polygon L_3 , which will characterize the simplest equation. This polygon should posses properties discussed above.

The fifth step. Selection of the simplest equation with unknown parameters that generates the polygon L_3 .

The sixth step. Determination of the undefined coefficients, which are present in the transformation (2.3) and in the simplest equation.

Remark 1. Very often suitable simplest equations can be found without making transformation (2.3). In this case (2.3) is an identity substitution and we set $y(z) \equiv Y(z)$, $L_2 \equiv L_1$.

Remark 2. In some cases the transformation can be included into the simplest equation. Then again (2.3) is an identity substitution.

Remark 3. The most powerful transformations, i.e. the transformations that generate new classes of exact solutions, are those that change the pole order of $y(z)$.

3 Exact solutions of the Kortevog – de Vries – Burgers equation

To demonstrate our method application let us find exact solutions of the Kortevog – de Vries – Burgers equation

$$u_t + uu_x + \beta u_{xxx} - \nu u_{xx} = 0. \quad (3.1)$$

It has the travelling wave reduction

$$u(x, t) = w(z), \quad z = x - C_0 t, \quad (3.2)$$

where $w(z)$ satisfies the equation

$$E[w] = \beta w_{zz} - \nu w_z + \frac{1}{2} w^2 - C_0 w + C_1 = 0. \quad (3.3)$$

Here C_0 , C_1 , α , and β are constants. In the case,

$$C_0 = -\frac{6\nu^2}{25\beta}, \quad C_1 = 0 \quad (3.4)$$

equation (3.3) is integrable and was solved by Painlevé [5, 6]. Its general solution is expressed via the Jacobi elliptic function. However equation (3.3) also describes a solitary wave. The solution in the form of this solitary wave was first obtained in [10] and later it was rediscovered a lot of times.

Let us formulate the following theorem.

Theorem 3.1. *Let $Y(z)$ be a solution of the equation*

$$Y_z + Y^2 - \frac{\nu^2}{10\beta} = 0. \quad (3.5)$$

Then

$$y(z) = C_0 + \frac{3\nu^2}{25\beta} - \frac{12\nu}{5}Y(z) - 12\beta Y(z)^2 \quad (3.6)$$

is a solution of the equation (3.3) in the case,

$$C_1 = \frac{C_0}{2} - \frac{18\nu^2}{625\beta} \quad (3.7)$$

Proof. At the first step we should find the Newton polygon L_1 corresponding to equation (3.3). For monomials of this equation we have points: $M_1 = (-2, 1)$, $M_2 = (-1, 1)$, $M_3 = (0, 2)$, $M_4 = (0, 1)$, $M_5 = (0, 0)$. The support of equation (3.3) is defined by four points: $Q_1 = M_1$, $Q_2 = M_3$, $Q_3 = M_5$ and $Q_4 = M_5$. Their convex hull is the triangle L_1 (Fig. 1). This triangle contains three vertexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3$) and three edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_3^{(1)} = [Q_1, Q_3]$.

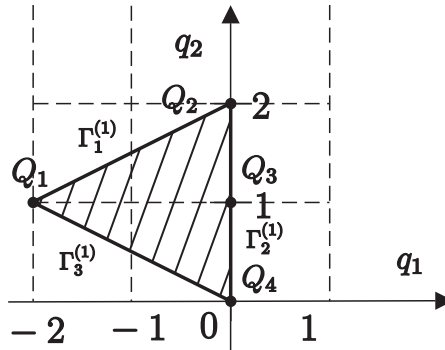


Figure 1: Polygon corresponding to the differential equation (3.3).

Solutions of equation (3.3) have the second – order singularity. Let us make the following transformation

$$w(z) = A_0 + A_1 Y(z) + A_2 Y(z)^2, \quad (3.8)$$

where $Y(z)$ satisfies the first – order differential equation and has the first – order singularity.

Substituting (3.8) into the equation (3.3), we obtain

$$\begin{aligned}
M_2[w[Y]] &= (\beta A_1 + 2\beta A_2 Y) Y_{zz} - (\nu A_1 + 2\nu A_2 Y) Y_z + \\
&+ 2\beta A_2 Y_z^2 + \left(A_0 A_2 - C_0 A_2 + \frac{1}{2} A_1^2 \right) Y^2 + A_1 A_2 Y^3 + \\
&+ \frac{1}{2} A_2^2 Y^4 + (A_0 A_1 - C_0 A_1) Y + \frac{1}{2} A_0^2 + C_1 - C_0 A_0 = 0.
\end{aligned} \tag{3.9}$$

The following points correspond to the monomials of this equation: $M_1 = (-2, 1)$, $M_2 = (-2, 2)$, $M_3 = (-2, 2)$, $M_4 = (-1, 1)$, $M_5 = (-1, 2)$, $M_6 = (0, 0)$, $M_7 = (0, 2)$, $M_8 = (0, 4)$, $M_9 = (0, 1)$, $M_{10} = (0, 2)$, $M_{11} = (0, 3)$, $M_{12} = (0, 0)$, $M_{13} = (0, 1)$, $M_{14} = (0, 2)$, $M_{15} = (0, 0)$. The support of equation (3.9) is determined by seven points: $Q_1 = M_1$, $Q_2 = M_2 = M_3$, $Q_3 = M_8$, $Q_4 = M_{11}$, $Q_5 = M_7 = M_{14}$ and $Q_6 = M_9 = M_{13}$. Their convex hull is the quadrangle (Fig. 2). This quadrangle has four vertexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) and four edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_3^{(1)} = [Q_3, Q_7]$, $\Gamma_4^{(1)} = [Q_1, Q_7]$. So we have constructed the Newton polygon L_2 .

Following our method we should find the Newton polygon L_3 with some part of edges parallel to those of L_2 . Besides that when the polygon L_3 moves along the plane, his vertexes should cover the support of equation (3.9).

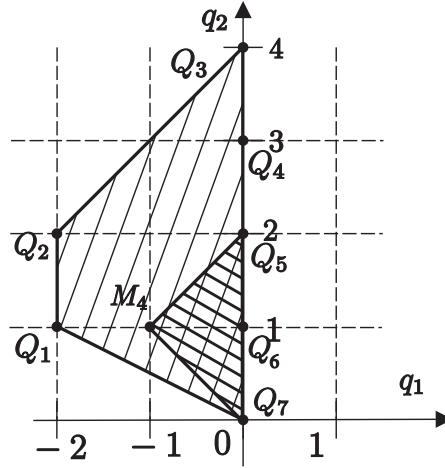


Figure 2: Polygons of the differential equations (3.9) and (3.10).

We see that the triangle with vertexes M_4 , Q_5 and Q_7 satisfy these requirements. Thus the simplest equation is the Riccati equation with constant coefficients

$$E_1[Y] = Y_z + Y^2 - b = 0. \tag{3.10}$$

Substituting

$$Y_z(z) = E_1[Y] - Y^2 + b \tag{3.11}$$

into equation (3.9) and equating coefficients at powers of $Y(z)$ to zero, yields algebraic equations for parameters A_2, A_1, A_0, b and C_1 . Solving these equations, we get

$$A_2 = -12\beta, \quad A_1 = -\frac{12\nu}{5}, \quad A_0 = C_0 + \frac{3\nu^2}{25\beta}, \quad b = \frac{\nu^2}{100\beta^2}, \quad (3.12)$$

$$C_1 = \frac{1}{2}C_0^2 - \frac{18\nu^4}{625\beta^2}$$

and the relation

$$M_2[w(Y)] = \hat{R} E_1[Y], \quad (3.13)$$

where \hat{R} is a differential operator. This completes the proof. \square

From equality (3.13) we see that if $Y(z)$ is a solution of equation (3.11), then $w(z)$ in formula (3.8) is a solution of equation (3.3).

Hence we have found the solitary wave in the form of a kink [10]

$$w(z) = C_0 + \frac{6\nu^2}{25\beta} - \frac{3\nu^2}{25\beta} \left(1 + \tanh \left\{ \pm \frac{\nu(x - C_0 t + \varphi_0)}{10\beta} \right\} \right)^2. \quad (3.14)$$

Here φ_0 is an arbitrary constant.

4 Exact solutions of the nonlinear fourth – order evolution equation

Let us look for exact solutions of the following fourth – order evolution equation [33]

$$u_t - 2u_x u_{xx} - u^2 u_{xx} - 2u u_x^2 + u_{xxxx} = 0. \quad (4.1)$$

Using the travelling wave reduction (3.2) and integrating with respect to z , we get

$$w_{zzz} - w_z^2 - w^2 w_z - C_0 w + C_1 = 0. \quad (4.2)$$

Later let us prove the following theorem.

Theorem 4.1. *Let $Y(z)$ be a solution of the equation*

$$Y_{zz} - Y Y_z - C_0 = 0. \quad (4.3)$$

Then $w(z) = Y(z)$ is a solution of the equation (4.2), provided that $C_1 = 0$.

Proof. Let us find the Newton polygon that corresponds to the equation (4.2). The following points are assigned to the monomials of this equation: $M_1 = (-3, 1)$, $M_2 = (-2, 2)$, $M_3 = (-1, 3)$, $M_4 = (0, 1)$, $M_5 = (0, 0)$. The carrier of the equation contains five points $Q_1 = M_1$, $Q_2 = M_2$, $Q_3 = M_3$, $Q_4 =$

M_4 $Q_5 = M_5$. Their convex hull is the quadrangle L_1 with four vertexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) and four edges $\Gamma_1^{(1)} = [Q_1, Q_3]$, $\Gamma_2^{(1)} = [Q_3, Q_4]$, $\Gamma_3^{(1)} = [Q_4, Q_5]$, $\Gamma_4^{(1)} = [Q_1, Q_5]$ (see Fig. 3). Solution of equation (4.2) have the first order pole. Suitable polygon L_3 we find without making the transformation. It is the triangle in Figure 3. Thus we set

$$w(z) \equiv Y(z), \quad (4.4)$$

where $Y(z)$ is a solution of the second order equation

$$E_2[Y] = Y_{zz} - aY Y_z - b = 0. \quad (4.5)$$

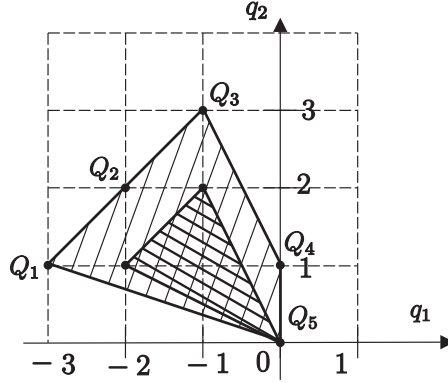


Figure 3: Polygons of the differential equations (4.2) and (4.5).

Substituting (4.4) and

$$Y_{zz} = E_2[Y] + aY Y_z + b \quad (4.6)$$

into equation (4.2) and equating coefficients at powers of $Y(z)$ to zero yields algebraic equations for parameters a , b and C_1 . Hence we get

$$a = 1, \quad b = C_0, \quad C_1 = 0. \quad (4.7)$$

We also obtain the relation

$$M_3[w(Y)] = \hat{R} E_2[Y], \quad (4.8)$$

where \hat{R} is a differential operator. This completes the proof. \square

Integrating equation (4.3) with respect to z yields the Riccati equation

$$Y_z - Y^2 - \frac{C_0}{2} z + 2C_3 = 0, \quad (4.9)$$

where C_3 is a constant of integration. Setting

$$Y(z) = -\frac{2\Psi_z}{\Psi} \quad (4.10)$$

in (4.9) we get

$$\Psi_{zz} + \left(\frac{C_0}{2} z - C_3 \right) \Psi = 0. \quad (4.11)$$

This equation is equivalent to the Airy equation. Its general solution is

$$\Psi(z) = C_1 Ai\{-2^{-1/3} C_0^{1/3} z + C_4\} + C_2 Bi\{-2^{-1/3} C_0^{1/3} z + C_4\}, \quad (4.12)$$

where $Ai(\zeta)$ and $Bi(\zeta)$ are the Airy functions and C_1 , C_2 and C_4 are arbitrary constants.

5 Exact solutions of the generalized Kuramoto – Sivashinsky equation

Let us look for exact solutions of the generalized Kuramoto – Sivashinsky equation, which can be written as

$$u_t + \alpha u^m u_x + \delta u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0. \quad (5.1)$$

Equation (5.1) at $m = 1$ is the famous Kuramoto – Sivashinsky equation [34, 35], which describes turbulent processes. Its exact solutions at $\beta = 0$ were first found in [34]. Its solitary waves at $\beta \neq 0$ were obtained in [10] and periodical solutions of this equation also at $\beta \neq 0$ were first presented in [13]. Recently it was shown [36, 37] that this equation did not have other solutions except found before.

Using the variables

$$x' = x \sqrt{\frac{\delta}{\gamma}}, \quad t' = t \frac{\delta^2}{\gamma}, \quad u' = u, \quad \sigma = \frac{\beta}{\sqrt{\gamma \delta}}, \quad \alpha' = \frac{\alpha \sqrt{\gamma \delta}}{\delta}, \quad (5.2)$$

we get the equation in the form (the primes are omitted)

$$u_t + \alpha u^m u_x + u_{xx} + \sigma u_{xxx} + u_{xxxx} = 0. \quad (5.3)$$

Using the travelling wave reduction

$$u(x, t) = y(z), \quad z = x - C_0 t \quad (5.4)$$

and integrating with respect to z , we get the equation

$$M_3[y] = y_{zzz} + \sigma y_{zz} + y_z - C_0 y + \frac{\alpha}{m+1} y^{m+1} = 0. \quad (5.5)$$

Here a constant of integration is equated to zero.

Let us present our result in the following theorem.

Theorem 5.1. Let $y(z)$ be a solution of the equation

$$y_z = \left(-\frac{9\alpha}{2m^3 + 11m^2 + 18m + 9} \right)^{\frac{1}{3}} y^{\frac{m+3}{3}} \mp \frac{3}{\sqrt{2m^2 + 18m + 27}} y. \quad (5.6)$$

Then $y(z)$ is also a solution of the equation (5.5), provided that $m \neq 0$, $m \neq -1$, $m \neq -\frac{3}{2}$, $m \neq -3$, $m \neq \frac{-9 \pm 3\sqrt{3}}{2}$, and

$$C_0 = \mp \frac{3(2m^2 + 9m + 9)}{(2m^2 + 18m + 27)^{3/2}}, \quad \sigma = \pm \frac{3(3+m)}{\sqrt{2m^2 + 18m + 27}}. \quad (5.7)$$

Proof. Monomials of equation (5.5) are determined by the following points: $M_1 = (-3, 1)$, $M_2 = (-2, 1)$, $M_3 = (-1, 1)$, $M_4 = (0, 1)$ and $M_5 = (0, m+1)$. The Newton polygon L_1 that corresponds to equation (5.5) is the major triangle in Figure 4 ((a): $m > 0$ and (b): $m < 0$). Suitable polygon L_3 can be found without making the transformation. It is the smaller triangle in Figure 4. This triangle contains the points: $Q_1 = (-1, 1)$, $Q_2 = M_4 = (0, 1)$, and $Q_3 = (0, 1 + \frac{m}{3})$.

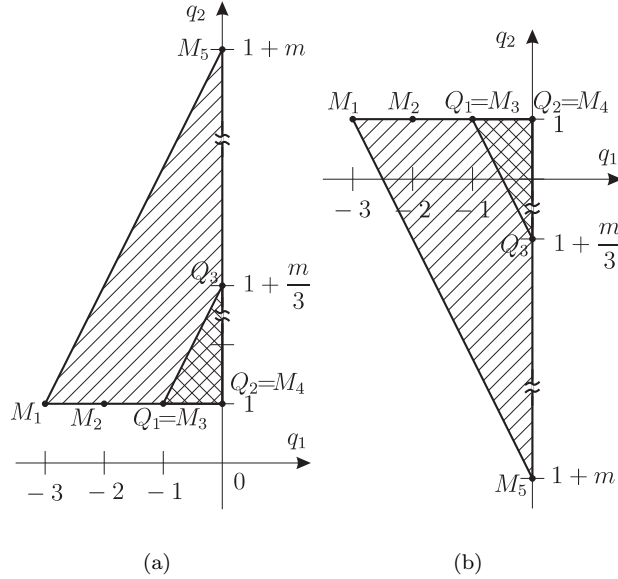


Figure 4: Polygons of the differential equations (5.5) and (5.8) at $m > 0$ (a) and $m < 0$ (b).

Consequently the simplest equation corresponding to L_3 can be written as

$$E_1[y] = y_z - A y^{\frac{m+3}{3}} - B y = 0, \quad (5.8)$$

where A and B are parameters to be found.

The general solution of (5.8) takes the form

$$y(z) = \left(C_1 \exp \left\{ -\frac{B m z}{3} \right\} - \frac{A}{B} \right)^{-\frac{3}{m}}. \quad (5.9)$$

Substituting

$$y_z = E_1[y] + A y^{\frac{m+3}{3}} + B y \quad (5.10)$$

into equation (5.5) and equating coefficients at powers of $y(z)$ to zero yields algebraic equations for parameters A , B , C_0 and σ in the form

$$A^3 (2m + 3)(m + 3)(m + 1) + 9\alpha = 0, \quad (5.11)$$

$$(m + 1)(B^3 + \sigma B^2 + B - C_0) = 0, \quad (5.12)$$

$$A^2 (3 + m)(m + 1)(Bm + 3B + \sigma) = 0, \quad (5.13)$$

$$(m + 1)(2\sigma^2 m^2 + 18\sigma^2 m + 27\sigma^2 - 9m^2 - 54m - 81) = 0. \quad (5.14)$$

Equations (5.11), (5.12), (5.13), and (5.14) can be solved and we obtain

$$A = (-9\alpha)^{1/3} (m^3 + 11m^2 + 18m + 9)^{-1/3} \quad (5.15)$$

(In final expressions we will take into consideration A_1 only.)

$$C_0 = B(B^2 + \sigma B + 1), \quad B = -\frac{\sigma}{3 + m}, \quad (5.16)$$

$$\sigma = \pm \frac{3(3 + m)}{\sqrt{2m^2 + 18m + 27}}. \quad (5.17)$$

So we have found parameters of the equation (5.9), conditions (5.7) and the relation

$$M_3[y] = \hat{R} E_1[y], \quad (5.18)$$

where \hat{R} is a differential operator. This completes the proof. \square

The general solution of equation (5.7) can be presented in the form

$$y(z) = \left(C + C_1 \exp \left\{ \pm \frac{mz}{\sqrt{2m^2 + 18m + 27}} \right\} \right)^{-\frac{3}{m}}, \quad (5.19)$$

where C_1 is an arbitrary constant and C is determined by expression

$$C = \pm \frac{\sqrt[3]{-9\alpha} \sqrt{2m^2 + 18m + 27}}{3\sqrt[3]{(m + 3)(2m^2 + 5m + 3)}}. \quad (5.20)$$

We would like to note that the value of solitary wave velocity (5.19) tends to $C_0 = \mp 1/\sqrt{27}$ as $m \rightarrow 0$, but at the same time $C_0 \rightarrow 0$ as $m \rightarrow \infty$.

Assuming $m = 1$ in (5.19) we obtain the known solitary wave

$$y(z) = \left(\frac{\sqrt{47} \sqrt[3]{-225\alpha}}{30} + C_1 \exp \left\{ \pm \frac{x - C_0 t}{\sqrt{47}} \right\} \right)^{-3}, \quad C_0 = \mp \frac{60}{47\sqrt{47}}. \quad (5.21)$$

and the value of the parameter σ : $\sigma = \pm 12/\sqrt{47}$. This solution of the Kuramoto – Sivashinsky equation is the kink [10–14].

6 Self similar solutions of the fifth – order Kortevg – de Vries equation

Let us find exact solutions of the fifth – order Kortevg – de Vries equation using our approach. This equation can be written as

$$u_t + u_{xxxxx} - 10 u u_{xxx} - 20 u_x u_{xx} + 30 u^2 u_x = 0. \quad (6.1)$$

It has the self similar solution

$$u(x, t) = \frac{1}{(5t)^{2/5}} y(z), \quad z = \frac{x}{(5t)^{1/5}}, \quad (6.2)$$

where $y(z)$ satisfies the nonlinear ODE of the form

$$y_{zzzzz} - 10 y y_{zzz} - 20 y_z y_{zz} + 30 y^2 y_z - z y_z - 2 y = 0 \quad (6.3)$$

Our results are summarized in the following theorem.

Theorem 6.1. *Let $Y(z)$ be a solution of the equation*

$$Y_{zzzzz} - 40 Y Y_{zz} Y_z - 10 Y^2 Y_{zzz} - 10 Y_z^3 + 30 Y^4 Y_z - Y - z Y_z = 0 \quad (6.4)$$

Then

$$y(z) = \pm Y_z - Y^2 \quad (6.5)$$

is a solution of equation (6.3).

Proof. The following points correspond to the monomials of this equation: $M_1 = (-5, 1)$, $M_2 = (-3, 2)$, $M_3 = (-3, 2)$, $M_4 = (-1, 3)$, $M_5 = (0, 1)$ and $M_6 = (0, 1)$. These points generate the support of equation (6.3): $Q_1 = M_1 = (-5, 1)$, $Q_2 = M_4 = (-1, 3)$, $Q_3 = M_5 = M_6 = (0, 1)$ and $Q_4 = M_2 = M_3 = (-3, 2)$. Now we can plot the Newton polygon L_1 (see Fig. 5)

Solutions of equation (6.3) have the second order pole. Thus we can make the following transformation

$$y = A_1 Y + A_2 Y^2 + A_3 Y_z, \quad (6.6)$$

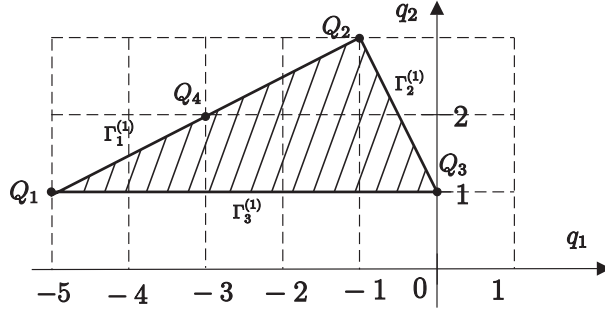


Figure 5: Polygon corresponding to the differential equation of (6.3).

where $Y(z)$ is a function of the first order pole. Substituting (6.6) into the equation studied, we obtain

$$\begin{aligned}
 & A_1 Y_{zzzzz} + A_2 Y_{zzzz}^2 + A_3 Y_{zzzzzz} - \\
 & -10 (A_1 Y + A_2 Y^2 + A_3 Y_z) (A_1 Y_{zzz} + A_2 Y_{zzz}^2 + A_3 Y_{zzzz}) - \\
 & -20 (A_1 Y_z + A_2 Y_z^2 + A_3 Y_{zz}) (A_1 Y_{zz} + A_2 Y_{zz}^2 + A_3 Y_{zzz}) + \\
 & +30 (A_1 Y + A_2 Y^2 + A_3 Y_z)^2 (A_1 Y_z + A_2 Y_z^2 + A_3 Y_{zz}) - \\
 & -z (A_1 Y_z + A_2 Y_z^2 + A_3 Y_{zz}) - 2 A_1 Y - 2 A_2 Y^2 - 2 A_3 Y_z = 0.
 \end{aligned} \tag{6.7}$$

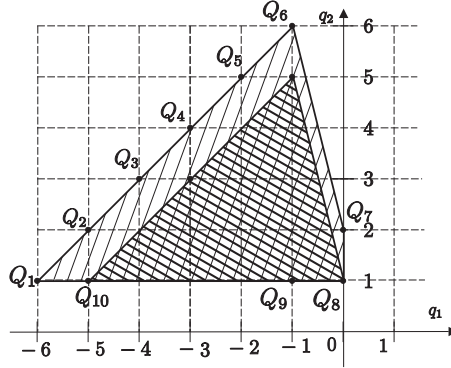


Figure 6: Polygons of the differential equations (6.7) and (6.8).

The support of this equation is defined by ten points: $Q_1 = (-6, 1)$, $Q_2 = (-5, 2)$, $Q_3 = (-4, 3)$, $Q_4 = (-3, 4)$, $Q_5 = (-2, 5)$, $Q_6 = (-1, 6)$, $Q_7 = (0, 2)$, $Q_8 = (0, 1)$, $Q_9 = (-1, 1)$, $Q_{10} = (-5, 1)$. In this case, the polygon L_2 is the quadrangle presented in Figure 6. Studying this polygon we can find the polygon L_3 for the simplest equation. It is the triangle also presented in Figure

6. Consequently the simplest equation for (6.7) is

$$E_5[Y] = Y_{zzzzz} + m_1 Y Y_{zz} Y_z + m_2 Y^2 Y_{zzz} + m_3 Y_z^3 +$$

$$+ m_4 Y^4 Y_z + m_5 Y + m_6 z Y_z = 0, \quad (6.8)$$

where m_1, m_2, m_3, m_4, m_5 and m_6 are unknown parameters to be found.

Substituting

$$Y_{zzzzz} = E_5[Y] - m_1 Y Y_{zz} Y_z - m_2 Y^2 Y_{zzz} - m_3 Y_z^3 -$$

$$- m_4 Y^4 Y_z - m_5 Y - m_6 z Y_z \quad (6.9)$$

into equation (6.7) and equating coefficients at different powers of $Y(z)$ to zero yields algebraic equations for parameters m_1, m_2, m_3, m_4, m_5 and m_6 . As a result we get

$$m_1 = -40, \quad m_2 = -10, \quad m_3 = -10, \quad m_4 = 30, \quad (6.10)$$

$$m_5 = -1, \quad m_6 = -1,$$

and the relation

$$M_5[y(Y)] = \hat{R} E_5[Y], \quad (6.11)$$

where \hat{R} is a differential operator. This completes the proof. \square

Equation (6.4) can be integrated in z and we get the fourth – order analogue to the second Painlevé equation

$$Y_{zzzz} - 10 Y^2 Y_{zz} - 10 Y Y_z^2 + 6 Y^5 - z Y - \beta = 0, \quad (6.12)$$

where β is an arbitrary constant. Thus we have expressed solutions of equation (6.3) (and consequently of equation (6.1)) through solutions of (6.12).

7 Self similar solutions of the fifth – order modified Kortevæg – de Vries equation

Let us find self similar solutions of the fifth – order modified Kortevæg – de Vries equation

$$u_t - 10u^2 u_{xxx} - 40u_x u_{xx} - 10u_x^3 + 30u^4 u_x + u_{xxxxx} = 0. \quad (7.1)$$

It has the self similar solution [38–40]

$$u(x, t) = (5t)^{-1/5} w(z), \quad z = x (5t)^{-1/5}, \quad (7.2)$$

where $w(z)$ satisfies

$$M_4[w] = w_{zzzz} - 10 w^2 w_{zz} - 10 w w_z^2 + 6 w^5 - z w - \beta = 0. \quad (7.3)$$

Let us present our result in the theorem.

Theorem 7.1. Let $Y(z)$ be a solution of the equation

$$Y_{zzz} + 2bY Y_{zz} - bY_z^2 - 6b^2Y^2Y_z - 3b^3Y^4 + \frac{z}{2b} = 0. \quad (7.4)$$

Then

$$w(z) = \pm bY(z) \quad (7.5)$$

is a solution of the equation (7.3), provided that $\beta = \mp 1/2$.

Proof. To begin with the Newton polygon corresponding to the equation (7.3) should be found. The following points: $M_1 = (-4, 1)$, $M_2 = (-2, 3)$, $M_3 = (-2, 3)$, $M_4 = (0, 5)$, $M_5 = (1, 1)$, $M_6 = (0, 0)$ are assigned to the monomials of the studied equation. The support of the equation contains five points $Q_1 = M_1$, $Q_2 = M_4$, $Q_3 = M_5$, $Q_4 = M_6$, $Q_5 = M_2 = M_3$. Their convex hull is the quadrangle L_1 with four vertexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) and four edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_3^{(1)} = [Q_3, Q_4]$, $\Gamma_4^{(1)} = [Q_1, Q_4]$ (fig. 7).

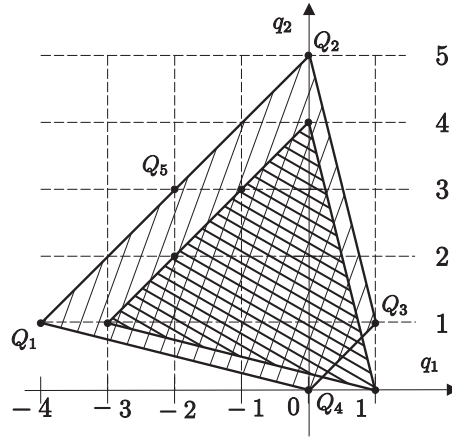


Figure 7: Polygons of the differential equations (7.3) and (7.8).

Leadings members of the equation (7.1) are located on the edge $\Gamma_1^{(1)} = [Q_1, Q_2]$. Substituting $w(z) = a_0 z^p$ into the equation

$$w_{zzzz} - 10w^2w_{zz} - 10ww_z^2 + 6w^5 = 0, \quad (7.6)$$

we get four families of power asymptotics for solutions of (7.3): $(p, a_0) = (-1, 1)$, $(-1, -2)$, $(-1, -3)$, and $(-1, 4)$. Again taking into account the pole order of solutions and the results of section 2, we can look for exact solutions of the equation (7.3) in the form

$$w(z) = A_1 Y(z), \quad A_1 \neq 0, \quad (7.7)$$

where $Y(z)$ is a solution of the following simplest equation

$$Y_{zzz} - aYY_{zz} - bY_z^2 - cY^2Y_z - dY^4 - mz = 0. \quad (7.8)$$

It is important to mention that the transformation (7.3) does not change the quadrangle L_1 and the Newton polygon L_3 for the simplest equation (7.7) possesses all necessary properties. In other words $L_1 \equiv L_2$ and L_3 is the triangle (see Fig. 7). The parameters a , b , c , d , and m can be found.

Substituting (7.7) and

$$Y_{zzz} = E_3[Y] + aYY_{zz} + bY_z^2 + cY^2Y_z + dY^4 + mz = 0 \quad (7.9)$$

into equation (7.3) and equating coefficients at different powers of $Y(z)$ to zero yields algebraic equations for parameters a , b , c , d , m , β and A_1 . Finally, we obtain

$$a = -2b, \quad c = 6b^2, \quad d = 3b^3, \quad m = -\frac{1}{2b}, \quad (7.10)$$

$$A_{1,2} = \pm b, \quad \beta = \mp \frac{1}{2}$$

and the relation [39]

$$M_4[w(Y)] = \hat{R}E_3[Y], \quad (7.11)$$

where \hat{R} is a differential operator. This completes the proof. \square

Consequently there exist solutions of the studied equation expressed through the function $Y(z)$ at $\beta = 1/2$ or $\beta = -1/2$. Let us show that solutions of the equation (7.4) are associated with solutions of the first Painlevé equation

$$v_{zz} = Cv^2 + Dz. \quad (7.12)$$

Making the transformation

$$v(z) = Aw^2 + Bw_z \quad (7.13)$$

in the equation (7.5), yields the equation

$$Bw_{zzz} + 2Aw_{zz} + (2A - CB^2)w_z^2 - 2CABw^2w_z - CA^2w^4 - Dz = 0. \quad (7.14)$$

We see that the Newton polygon corresponding to this equation coincides with the triangle L_3 at Figure 6. Combining (7.4) and (7.10), we obtain

$$w(z) \equiv Y(z), \quad (7.15)$$

provided that $A = b$, $B = 1$, $C = 3b$, and $D = -1/2b$. In other words, if $Y(z)$ is a solution of (7.4), then $v(z) = bY^2 + Y_z$ is a solution of

$$v_{zz} = 3bv^2 - \frac{1}{2b}z. \quad (7.16)$$

We would like to mention that without loss of generality it can be set $b = \pm 1$.

8 Solitary waves of the sixth order nonlinear evolution equation

Let us find exact solutions of the sixth order nonlinear evolution equation that is applied at description of turbulent processes [41,42]

$$u_t + uu_x + \beta u_{xx} + \delta u_{xxx} + \varepsilon u_{xxxx} = 0. \quad (8.1)$$

It is known that Kuramoto – Sivashinskiy equation and the Ginzburg – Landau equation that are used at description of turbulence are non-integrable ones because they do not pass the Painlevé test [5]. However, these equations have a list of special solutions [5,10,13,36,38].

Equation (8.1) does not pass the Painlevé test as well and thus this equation is also non-integrable. However one can expect that equation (8.1) has some special solutions. This equation admits the travelling waves reduction

$$u(x,t) = y(z), \quad z = x - C_0 t, \quad (8.2)$$

where $y(z)$ satisfies the equation

$$C_1 - C_0 y + \frac{1}{2} y^2 + \beta y_z + \delta y_{zzz} + \varepsilon y_{zzzz} = 0. \quad (8.3)$$

C_1 is a constant of integration. As a result we get the following theorem.

Theorem 8.1. *Let $Y(z)$ be a solution of the equation*

$$Y_z + Y^2 - \alpha_k = 0, \quad (k = 1, 2, \dots, 6). \quad (8.4)$$

Then

$$y(z) = 30240 \varepsilon Y^5 + \left(\frac{2520}{11} \delta - 50400 \varepsilon \alpha_k \right) Y^3 + \left(-\frac{2520}{11} \delta \alpha_k + 20160 \varepsilon \alpha_k^2 + \frac{1260}{251} \beta - \frac{12600}{30371} \frac{\delta^2}{\varepsilon} \right) Y + C_0 \quad (8.5)$$

is a solution of the equation (8.3) provided that

$$C_1 = \frac{4112640}{11} \frac{\delta^5 w_k^4}{\varepsilon^3} - 9999360 \frac{\delta^5 w_k^5}{\varepsilon^3} - \frac{5080320}{251} \frac{\delta^3 w_k^3 \beta}{\varepsilon^2} - \frac{55460160}{30371} \frac{\delta^5 w_k^3}{\varepsilon^3} + \frac{660240}{2761} \frac{\delta^3 w_k^2 \beta}{\varepsilon^2} - \frac{25200}{30371} \frac{\delta^5 w_k^2}{\varepsilon^3} - \frac{1260}{251} \frac{\beta^2 \delta w_k}{\varepsilon} + \frac{12600}{30371} \frac{\beta \delta^3 w_k}{\varepsilon^2} + \frac{1}{2} C_0^2, \quad (8.6)$$

$$\beta = -\frac{213811840 \varepsilon^3 \alpha_k^3 - 10204656 \delta \varepsilon^2 \alpha_k^2 - 2045 \delta^3 - 92400 \delta^2 \varepsilon \alpha_k}{121 \varepsilon (9240 \varepsilon \alpha_k + 79 \delta)}, \quad (8.7)$$

where α_k and w_k are found from

$$\alpha_k = \frac{\delta w_k}{\varepsilon} \quad (k = 1, 2, \dots, 6), \quad (8.8)$$

$$w_1 = -\frac{1}{220}, \quad w_2 = -\frac{5}{176}, \quad w_3 = -\frac{1}{440}, \quad (8.9)$$

$$w_4 = \frac{1}{52800} \left(557 - \frac{46031}{m} + m \right), \quad (8.10)$$

$$w_{5,6} = \frac{1}{52800} \left(\frac{46031}{2m} - \frac{m}{2} + 557 \pm \frac{i\sqrt{3}}{2} \left(m + \frac{46031}{m} \right) \right), \quad (8.11)$$

$$m = (113816753 + 1260\sqrt{8221079733})^{\frac{1}{3}} \approx 610,966. \quad (8.12)$$

Proof. The following points correspond to the monomials of equation (8.3): $Q_1 = (-5, 1)$, $Q_2 = (0, 2)$, $Q_3 = (0, 0)$, $Q_4 = (-3, 1)$, and $Q_5 = (-1, 1)$. The Newton polygon L_1 of the equation is the triangle (Fig. 8).

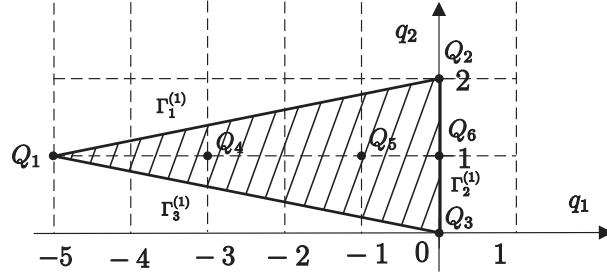


Figure 8: Polygon corresponding to the differential equation (8.3).

Solutions of equation (8.3) have the fifth – order pole. Let us express them through a function $Y(z)$ possessing the first – order pole

$$y(z) = A_0 + A_1Y + A_2Y^2 + A_3Y^3 + A_4Y^4 + A_5Y^5. \quad (8.13)$$

Substituting transformation (8.5) into (8.3), we get new equation and the corresponding polygon L_2 in the form of quadrangle (Fig.9). One of the suitable polygons L_3 is the triangle in Fig. 9. This triangle is assigned to the Riccati equation with constant coefficients

$$E_1[Y] = Y_z + Y^2 - \alpha = 0. \quad (8.14)$$

Substituting

$$Y_z = E_1[Y] - Y^2 + \alpha_k \quad (8.15)$$

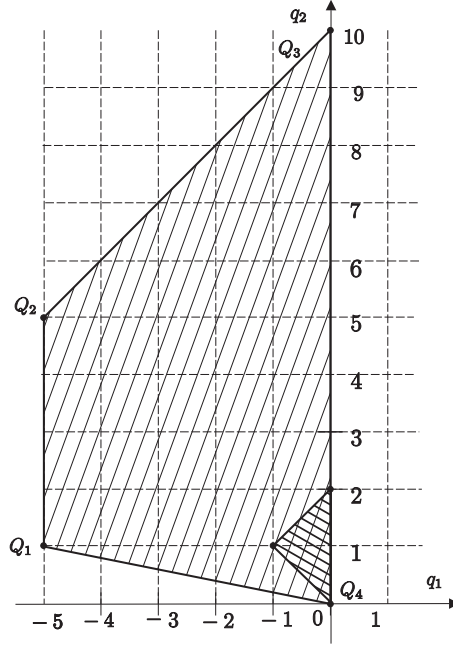


Figure 9: Polygon corresponding to the transformed differential equation (8.3) and to the simplest differential equation (8.14).

into equation (8.4) and equating coefficients at different powers of $Y(z)$ to zero yields algebraic equations for parameters $A_5, A_4, A_3, A_2, A_1, A_0, C_1, \beta, \alpha_k$ and w_k . Solving these equations we have

$$A_5 = 30240 \varepsilon, \quad A_4 = 0, \quad A_3 = \frac{2520\delta}{11} - 50400\varepsilon\alpha, \quad A_2 = 0, \quad (8.16)$$

$$A_1 = -\frac{2520}{11} \delta \alpha + 20160 \varepsilon \alpha^2 + \frac{1260}{251} \beta - \frac{12600}{30371} \frac{\delta^2}{\varepsilon}, \quad A_0 = C_0.$$

For the parameters C_1, β, α_k , and w_k we obtain expressions (8.6), (8.7), (8.8), (8.9), (8.10), and (8.12). We also see that the following relation

$$M_5[y(Y)] = \hat{R} E_1[Y], \quad (8.17)$$

holds, where \hat{R} is a differential operator. This completes the proof. \square

The general solution of equation (8.4) is the following

$$Y(z) = \sqrt{\alpha_k} \tanh(\sqrt{\alpha_k} z + \varphi_0), \quad (k = 1, 2, \dots, 6). \quad (8.18)$$

Substituting (8.18) into (8.5) and taking into account that $\alpha_k = w_k \delta / \varepsilon$ ($k = 1, \dots, 6$) we get six different solitary waves of equation (8.3).

9 Exact periodic solutions of equation (8.1)

It was mentioned in the previous section that solutions of equation (8.3) have the fifth order pole. Thus let us make the following transformation

$$y(z) = B_1 + B_2 R(z) + B_3 R_z + B_4 R^2 + B_5 R R_z, \quad (9.1)$$

where B_k ($k = 1, \dots, 5$) are constants and $R = R(z)$ is a function of the second order pole. Now let us prove the following theorem.

Theorem 9.1. *Let $R(z)$ be a solution of the equation*

$$R_z^2 = -2R^3 + aR^2 - \frac{1}{6}a^2R + \frac{1}{726}\frac{R\delta^2}{\varepsilon^2} \pm \frac{1}{2541}\frac{R\sqrt{21}\delta^2}{\varepsilon^2} + \quad (9.2)$$

$$\frac{1}{108}a^3 + \frac{13}{359370}\frac{\delta^3}{\varepsilon^3} \pm \frac{1}{119790}\frac{\sqrt{21}\delta^3}{\varepsilon^3} - \frac{1}{4356}\frac{a\delta^2}{\varepsilon^2} \mp \frac{1}{15246}\frac{a\sqrt{21}\delta^2}{\varepsilon^2}.$$

Then

$$y(z) = C_0 + 630 \left(\varepsilon a + \frac{\delta}{11} - 6\varepsilon R \right) R_z \quad (9.3)$$

is a solution of equation (8.1), provided that

$$\beta = \frac{10}{121}\frac{\delta^2}{\varepsilon}, \quad (9.4)$$

$$C_1^{(1,2)} = -\frac{10854}{161051}\frac{\delta^5}{\varepsilon^3} + \frac{1}{2}C_0^2 \mp \frac{2484}{161051}\frac{\sqrt{21}\delta^5}{\varepsilon^3}. \quad (9.5)$$

Proof. Substituting (9.1) into equation (8.3) we get a new equation of the sixth order. The Newton polygon L_2 of this equation is presented in Figure 10. Again we should construct suitable polygon L_3 . Let us take the triangle corresponding to the equation for the elliptic function

$$E_1[R] = R_z^2 + 2R^3 - aR^2 - 2bR - d = 0. \quad (9.6)$$

This triangle satisfies necessary requirements. Substituting

$$R_z^2 = E_1[R] - 2R^3 + aR^2 + 2bR + d, \quad (9.7)$$

$$R_{zz} = \frac{1}{2R_z} \frac{\partial E_{1,z}[R]}{\partial z} - 3R^2 + aR + b \quad (9.8)$$

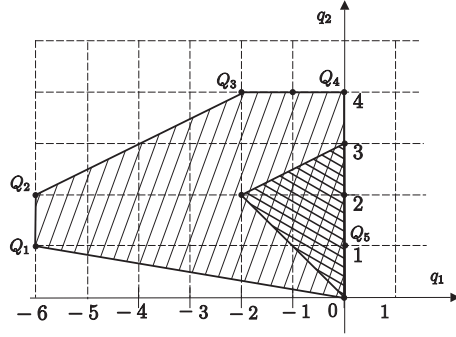


Figure 10: Polygons corresponding to the transformed differential equation (8.3) and to the simplest differential equation (9.6).

into equation (8.3) and equating coefficients at different powers of $R(z)$ and R_z to zero yields algebraic equations for the coefficients B_5 , B_4 , B_3 , B_2 , B_1 , and B_0 . As a result we obtain the following expressions

$$B_5 = -3780 \varepsilon, \quad B_4 = 0, \quad B_3 = 630 \varepsilon a + \frac{630}{11} \delta, \quad (9.9)$$

$$B_2 = 0, \quad B_1 = C_0,$$

$$\beta = \frac{10}{121} \frac{\delta^2}{\varepsilon}, \quad (9.10)$$

$$b_{1,2} = -\frac{1}{12} a^2 + \frac{1}{1452} \frac{\delta^2}{\varepsilon^2} \pm \frac{1}{5082} \frac{\sqrt{21} \delta^2}{\varepsilon^2}, \quad (9.11)$$

$$d_{1,2} = \frac{1}{108} a^3 + \frac{13}{359370} \frac{\delta^3}{\varepsilon^3} \pm \frac{1}{119790} \frac{\sqrt{21} \delta^3}{\varepsilon^3} - \quad (9.12)$$

$$-\frac{1}{4356} \frac{a \delta^2}{\varepsilon^2} \mp \frac{1}{15246} \frac{a \sqrt{21} \delta^2}{\varepsilon^2},$$

and the relation

$$M_6[y(R)] = \hat{R} E_1[R], \quad (9.13)$$

where \hat{R} is a differential operator. This completes the proof. \square

If R_1, R_2 , and R_3 such that $R_1 \geq R_2 \geq R_3$ are real roots of the equations

$$2R^3 - aR^2 + \left(\frac{1}{6}a^2 - \frac{\delta^2}{726\varepsilon^2} \mp \frac{\delta^2\sqrt{21}}{2541\varepsilon^2} \right) R - \frac{1}{108}a^3 -$$

$$- \frac{13}{359379} \frac{\delta^3}{\varepsilon^3} \mp \frac{\delta^3\sqrt{21}}{119790\varepsilon^3} + \frac{a\delta^2}{4356\varepsilon^2} \pm \frac{a\delta^2\sqrt{21}}{15246\varepsilon^2} = 0,$$
(9.14)

then the general solution of (9.2) can be written as

$$R(z) = R_2 + (R_1 - R_2) \operatorname{cn}^2\left(z\sqrt{\frac{R_1 - R_3}{2}}, S\right), \quad S^2 = \frac{R_1 - R_2}{R_1 - R_3}. \quad (9.15)$$

Thus we have found several solutions of equation (8.1) at different values of the parameters. These solution are solitary and periodic waves (see expressions (8.5) and (9.3)).

10 Conclusion

In this paper a new method for finding exact solutions of nonlinear differential equations is presented. Our aim was to express solutions of the equation studied through solutions of the simplest equations. To search the simplest equation we applied the power geometry and constructed the Newton polygons of nonlinear differential equations. The method is more powerful than many other methods because the structure of a solution is not fixed but can be found using the stated algorithm. Thus one can obtain quite general classes of exact solutions. It is important to mention that transformations between nonlinear differential equations can be also found with a help of our approach. As an example of our method application exact solutions of some nonlinear differential equations were found. In particular, solutions of the fourth – order evolution equation (4.1) were expressed in terms of the Airy functions; the connection between self – similar solutions of the fifth – order Kortevog – de Vries equation (6.1) and the fifth – order modified Kortevog – de Vries equation (7.1) were constructed; one – parametric family of exact solutions for the generalized Kuramoto - Sivashinsky equation (5.1) was found. Besides that we obtained solitary waves and periodic solutions of the sixth – order evolution equation (8.1) that is used at description of turbulence [42, 43].

11 Acknowledgments

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