

On exclusion type inhomogeneous interacting particle systems

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Abstract

For a large class of *inhomogeneous* interacting particle systems (IPS) on a lattice we develop a rigorous method for mapping them onto *homogeneous* IPS. Our novel approach provides a *direct* way of obtaining the statistical properties of such inhomogeneous systems by studying the far simpler homogeneous systems. In the cases when the latter can be solved exactly our method yields an exact solution for the statistical properties of an inhomogeneous IPS. This approach is illustrated by studies of three of IPS, namely those with particles of different sizes, or with varying (between particles) maximal velocities, or accelerations.

Key words: interacting particle system, exclusion process, parallel dynamics, traffic flow model.

1 Introduction

A very substantial progress on the understanding of statistical properties of lattice interacting particle systems (IPS) (see an excellent review in [5] and numerous references therein) has been achieved mainly for continuous time systems describing interactions of identical particles. Only recently results related to non homogeneous particle systems started to show up. Some of them analyze the presence of a single inhomogeneity (like a street light in [4]) or spatially varying hopping rates (see, e.g. [7, 6]), while in some other papers the situation when a single particle occupies several lattice sites were considered (see, e.g. [3]). In each of these papers the authors developed new (and quite complicated) constructions or approximations to deal with the inhomogeneities. Our strategy is based on a completely different idea, namely we reduce the analysis of an inhomogeneous problem to a homogeneous one for which a solution is much simpler or even already known. Returning to the original setting one is able to recover the complete statistical description for the inhomogeneous system. In distinction to complicated mean field approximations the exact constructions that we use are surprisingly simple and straightforward. We study three new situations when the analysis of an inhomogeneous particle system can be reduced to a homogeneous one. The first of them is the case when the particles differ in size, i.e. each particle occupies several lattice sites and, what is more important, the sizes of different particles might differ. It is worth note that the ability to deal with particles of different sizes is very important from the point of view of applications to dynamics of traffic flows (see a review with numerous references in [8]), where ordinary vehicles and buses or trucks are clearly of different sizes, or to various biological models like ion channels and mRNA translation where ribosomes and large molecules or vesicles might be of very different lengths. The only known result related to such systems [3] describing the motion of identical ‘long’ particles is based on a mean field approximation. Note also that the difference in particle sizes represents a fundamental obstacle for the application of one of the basic tools of the IPS theory – the coupling method.

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As we shall show in Section 2 due to a reduction to a homogeneous system with particles of unit size the exact solution is readily available. The main result of this Section is that if the only difference between the particles in the system is their size then there exists a bijection (one-to-one correspondence) between the original system and the system with identical particles having the same other properties. The idea here to ‘compress’ ‘long’ particles into ‘short’ ones cutting the ‘odd’ lattice sites. It might be surprising that details of the dynamics are not important and the system might be both probabilistic or deterministic. However, our construction holds only in the case of the one-dimensional lattice and an exclusion type constraint: not more than one particle may be present at a lattice site. At present it is not clear if there are any possible generalizations to the multidimensional case.

In Section 3 we consider a version of the model introduced in Section 2 in which in distinction to the previous setting we assume that each particle has its own maximal velocity but all of them have the same unit size. In particular this setting may appear as a result of the reduction of particle sizes by a procedure discussed in Section 2. Here in order to homogenize the system we apply a kind of a substitution dynamics.

In Section 4 we consider yet another generalization of the well-known Nagel-Schreckenberg (NS) traffic model [9] based on the introduction of a fractional acceleration of particles made in [2]. Recall that the original NS model already contained the acceleration term which was assumed to be an integer. In [2] it has been shown that the presence of the fractional acceleration leads to very rich hysteresis type phenomena. Considered from a bit more general point of view this model describes the motion of particles on a lattice under the action of a constant force represented by the acceleration. Therefore if the particles in the configuration have different masses or sizes it is natural to think that under the action of the same force they will get different accelerations. This is exactly the case we shall consider here. Again the different accelerations might be the result of the reduction of particle sizes.

2 Dynamics of different size particles systems

Consider a class of locally interacting particle systems on the one-dimensional integer lattice \mathbb{Z} . Each particle is described by its *position* – the most left site i which it occupies on the lattice, its *length* ℓ describing the number of lattice sites it occupies, its *velocity* v and may be some other parameters. The dynamics is defined as follows. At each site of the lattice there is an alarm-clock and at time $t > 0$ we consider only those particles which occupy lattice sites where the alarm rings. For each such particle with the position $i \in \mathbb{Z}$ we calculate its new velocity v_i using a (random or deterministic) procedure which is the same for all particles and does not depend neither on time nor on the other particles in the configuration. About the velocity we shall assume only that $|v_i| \leq V_{\max} < \infty$. Here V_{\max} plays the role of the largest allowed velocity and its boundedness defines the locality of interactions.

Then one checks a certain *admissibility* condition related to the possibility to move a particle from the site i to the site $i + v_i$. We assume that the admissibility condition is again local and depends only on the present positions of the particles in a $2V_{\max}$ lattice neighborhood of the site i . A natural assumption here is that the velocity v_i is not admissible if during the movement from the site i to the site $i + v_i$ the particle needs to go through an occupied site. Only if the admissibility condition is satisfied the particle is moved to a new position. Then for all sites to where the particles were moved we restart the alarm-clocks (again using a certain random or deterministic procedure).

We assume that the procedures used to choose new velocities and to restart the alarm-clocks are the same for all sites and do not depend on time.

Depending on the way how one restarts the alarm-clocks both continuous and discrete time particle systems can be considered. In what follows we restrict ourselves to a (more interesting

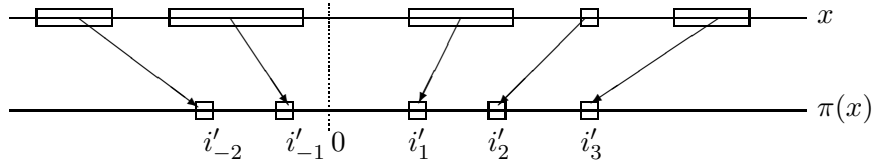


Figure 1: Mapping of an inhomogeneous configuration to a homogeneous one.

from our point of view and much less studied) discrete time case, assuming that the alarm-clocks start with the same setting and after each restart we add one to the time. Therefore all particles are trying to move simultaneously.

A typical and well-known model satisfying our assumptions is the so called exclusion process (see e.g. [5]). A particular deterministic case well suited for the description of traffic flows will be studied in Section 4.

The scheme of the size reduction is as follows: we introduce a very general dynamics of particles on an integer lattice where the next position/velocity of a particle depends only on the present position/velocity and on the positions of the particles in its neighborhood. Then we construct a bijective map π from the space of original particle configurations to the space of configurations of equal size particles which induces the new dynamics. The idea is to map simultaneously each original particle to a particle of size one and to delete all sites occupied by the particle except for just one of them from the lattice. To make the formal definition of the application of the map π to a configuration $x \in \mathbb{X}$ we enumerate all particles in this configuration according to their natural order by integers and setting the index 0 to the particle occupying the smallest nonnegative position on the lattice \mathbb{Z} . Thus for each index $j \in \mathbb{Z}$ we know the position i_j and the length ℓ_j of the corresponding particle. The positions of unit size particles in the configuration $\pi(x)$ we define recursively. First we put a particle of unit size to the position i_0 . After that we choose the particle with the index $j = 1$ in the original configuration and put a particle of unit size to the position $i_1 - \ell_0 + 1$. The next particle to the right in the original configuration corresponds to the index $j = 2$ and will go to the position $i_2 - \ell_0 + 1 - \ell_1$, etc. Thus the particle with the index $j > 0$ gets the position

$$i'_j := i_j - \sum_{k=0}^{j-1} (\ell_k - 1).$$

Similarly one defines positions for the particles with negative indices: if $j < 0$ then

$$i'_j := i_j + \sum_{k=j}^{-1} (\ell_k - 1).$$

Eventually we define the configuration $\pi(x)$ such that for each $j \in \mathbb{Z}$ there is a particle of unit size at site i'_j and all other sites are occupied by vacancies (see Fig 1).

Observe that the enumeration is preserved under the action of the map π and we assume that all parameters of the particles except their sizes are preserved as well. Note also that the enumeration makes the map π invertible, so this is a bijection. Nevertheless the total number of sites in a finite segment of the lattice may become smaller after the action of the map π .

Since in the model under consideration the dynamics does not depend on the sizes of the particles, the dynamics of the system after the application of the map π is preserved. Therefore if we know any statistical description of the new homogeneous particle system (e.g., its invariant measure, correlation functions, etc.) then applying the inverse map π^{-1} we get immediately the corresponding description of the original system.

Despite giving the complete information about the conjugation between the original inhomogeneous model and the resulting homogeneous one the map π is very complicated and it is desirable to have a more direct way to derive relations between various statistics of the original and the resulting systems. As we shall show at least for some important statistics, such as limit average particle velocities, the presence of the bijection gives a possibility to calculate them rigorously using only the average size of particles and their density.

Denote the space of positions corresponding to particles in admissible particle configurations by $\mathbb{X} \subset \{0, 1\}^{\mathbb{Z}}$. There is an important property that holds for all systems we consider here: particle conservation. For a configuration $x \in \mathbb{X} := \{0, 1\}^{\mathbb{Z}}$ and a finite subset $I \subset \mathbb{Z}$ denote by $\rho(x, I)$ the number of particles from the configuration x located in I divided by the total number of sites in I . Clearly $0 \leq \rho(x, I) \leq 1$. Choosing a sequence of lattice segments I_n of length n we consider the limit $\lim_{n \rightarrow \infty} \rho(x, I_n)$. If this limit exists and does not depend on the sequence $\{I_n\}_n$ we call it the *particle density* of the configuration $x \in \mathbb{X}$ and denote by $\rho(x)$. To show that the particle density is conserved under dynamics consider a segment of sites I of length L and denote by N_p and N'_p the numbers of particles in this segment at time t and by $t + 1$. During one time step at most two particles may leave or enter the segment I and thus $|N_p - N'_p| \leq 2$. Therefore passing to the limit as $L \rightarrow \infty$ we get the conservation of the density.

To this end let us introduce the notion of the *average velocity* of a particle. Let $L(x, i, t)$ be the distance covered during the time t by a particle in the configuration x located initially (at $t = 0$) at the site $i \in \mathbb{Z}$. Then by the average velocity of this particle we mean

$$V(x, i, t) := \frac{1}{t} L(x, i, t)$$

and consider also the limit average velocity

$$V(x, i) := \lim_{t \rightarrow \infty} V(x, i, t)$$

provided that it is well defined, otherwise one considers limit points of the sequence $\{V(x, i, t)\}_t$. In Section 4 we shall show that under some natural assumptions the above limit is well defined and is the same for all particles. In general this might not be the case, nevertheless we shall show that all limit points of the average velocities in the original system can be easily calculated from the corresponding limit points obtained for the homogeneous model.

To simplify notation we shall use the sign “prime” for various parameters of particles related to the configuration $x' := \pi(x)$. Choose a particle in the configuration x having a position $i \in \mathbb{Z}$ and compare the distance $L(x, i, t)$ which it covers during the time $t > 0$ to the distance $L'(x', i', t)$ covered by the corresponding particle located initially at $i' \in \mathbb{Z}$ in the homogeneous model. For a positive integer ℓ' denote by $N_p(\ell')$ and $N_v(\ell')$ the number of particles and vacancies respectively in the segment $[i', i' + \ell' - 1]$ of the configuration x' . Set

$$s(\ell') := (\ell' - N_v(\ell')) / N_p(\ell')$$

and assume that the limits

$$s := \lim_{\ell' \rightarrow \infty} s(\ell'), \quad \rho := 1 - \lim_{\ell' \rightarrow \infty} N_v(\ell') / \ell', \quad \rho' := \lim_{\ell' \rightarrow \infty} N_p(\ell') / \ell'$$

do exist. The first of them is the average size of a particle in the original configuration and the other two are the densities of particles in the original and the homogeneous models. Therefore the limits certainly exist if the configuration x' has the particle density. Note that in the definition of the value ρ the denominator is equal to ℓ (corresponding to the inhomogeneous system) rather than ℓ' .

Let $V'(x', i')$ be a limit point of the average velocities in the homogeneous system, i.e. there exists a sequence of moments of time $t_k \xrightarrow{k \rightarrow \infty} \infty$ such that

$$V'(x', i') := \lim_{k \rightarrow \infty} L'(x', i', t_k)/t_k.$$

For any $t > 0$ one has

$$L(x, i, t) = s(L'(x', i', t)) \cdot N_p(L'(x', i', t)) + N_v(L'(x', i', t))$$

while

$$L'(x', i', t) = N_p(x', i', t) + N_v(x', i', t).$$

Thus

$$L(x, i, t) - L'(x', i', t) = (s(L'(x', i', t)) - 1) \cdot N_p(L'(x', i', t)).$$

Therefore

$$\frac{L(x, i, t_k)}{t_k} = \frac{L'(x', i', t_k)}{t_k} \cdot \frac{L(x, i, t_k)}{L'(x', i', t_k)} \xrightarrow{k \rightarrow \infty} (1 + (s - 1)\rho') \cdot V'(x', i').$$

Hence passing the average velocities to the limit along the same sequence of moments of time $\{t_k\}$ we get

$$V(x, i) = (1 + (s - 1)\rho') \cdot V'(x', i'). \quad (2.1)$$

Similarly, but much simpler one gets the relation between the particle densities in the original and the corresponding homogeneous systems:

$$\rho := 1 - \lim_{\ell \rightarrow \infty} \frac{N_v(\ell')}{\ell} = 1 - (1 - \rho') \cdot \lim_{\ell \rightarrow \infty} \frac{\ell'}{\ell} = \frac{s\rho'}{1 + (s - 1)\rho'}. \quad (2.2)$$

Note that both these results do not depend neither on the fine features of the dynamics under consideration nor on the details of the distribution of particle sizes. Observe also that a naive idea to construct the relation between the average velocity and the particle density in the original system in terms of the density of the occupied sites using the corresponding formulae known for the homogeneous system does not work.

3 Different maximal velocities

In the model we have discussed in the previous section there was a parameter V_{\max} describing the maximal available velocity of particles. Assume now that this parameter varies from one particle to another but each velocity may take only two values: 0 or the corresponding positive maximal value. Our aim is to show that one can construct a new bijection \mathcal{C} between the inhomogeneous system with particles having different velocities and a homogeneous one having only identical ‘slow’ particles having the unit maximal velocities.

This can be done as follows:

$$\dots 0001_2 0000001_1 001_4 0000000 \dots \rightarrow \dots 0_3 1_2 0_2 0_4 1_1 0_1 0_1 1_4 0_4 0_3 \dots$$

Here in the left representation 0 stands for a vacancy and 1 with and the index v denotes a particle with the maximal available velocity v . Under the dynamics a particle “exchanges” its position with a certain number of succeeding vacancies. Therefore in the second representation we code both particles and vacancies by groups represented by 0_v and 1_v . The indices corresponding to vacancies play here a different role: a zero with an index v means v vacancies in the original

configuration. The index of 0 is the minimum between the index the preceding 1 and the total number of vacancies in the original configuration immediately after the particle, i.e.

$$\dots 1_2 0001_3 01_1 0000\dots \rightarrow \dots 1_2 0_2 1_3 0_1 1_1 0_1 0_3 \dots$$

The dynamics of the new system again consists of exchanges of particles and vacancies.

To make the above coding precise we introduce the alphabet $\mathcal{A} := \{0_1, 0_2, \dots, 0_{V_{\max}}, 1_0, 1_1, \dots, 1_{V_{\max}}\}$ with $2V_{\max} + 1$ elements and a map $\mathcal{C} : X \rightarrow \mathcal{A}^{\mathbb{Z}}$ defined through the following system of substitutions:

$$\begin{aligned} 1_v \underbrace{0 \dots 0}_{n} 1_{v'} &\xrightarrow{\mathcal{C}} 1_v \underbrace{0_v \dots 0_v}_{\lfloor n/v \rfloor} 0_{n - \lfloor n/v \rfloor v} 1_{v'} \\ \underbrace{0 \dots 0}_{\infty} 1_v &\xrightarrow{\mathcal{C}} \underbrace{0_v \dots 0_v}_{\infty} 1_v & 1_v \underbrace{0 \dots 0}_{\infty} &\xrightarrow{\mathcal{C}} 1_v \underbrace{0_v \dots 0_v}_{\infty}. \end{aligned}$$

Here v is the maximal velocity of the particle immediately preceding the block of zeros and $\lfloor \cdot \rfloor$ stands for the integer part of a number. The last two relations define the action of \mathcal{C} on ‘tails’ of x consisting entirely of vacancies. In other words a configuration is divided into blocks of consecutive vacancies surrounded by particles and each of the blocks is substituted by a block of zeroes indexed by the maximal velocity of the particle immediately preceding this block (except for the last indexed zero where the index is calculated as the remainder) according to the above substitution rules.

As we see the dynamics of the original system is equivalent through the bijection \mathcal{C} to the dynamics of the composition of the dynamics with ‘slow’ particles composed with \mathcal{C} . Using the approach similar to the one developed in the previous Section one can calculate explicit relations between the limit points of average velocities in the original system and in the constructed homogeneous one.

Note that a similar idea of the substitution dynamics has been applied earlier in another author’s paper [1] to reduce the analysis of a deterministic homogeneous model of a traffic flow with ‘fast’ particles to the ‘slow’ particles case.

4 A traffic model with different particle accelerations

Let x be a configuration of particles on the one-dimensional integer lattice \mathbb{Z} having at most one particle at a site. To each particle we associate two real variables: $0 \leq v \leq V_{\max} < \infty$ (which we call *velocity*) and $0 \leq a \leq 1$ (which we call *acceleration*). A configuration is called *admissible* if each particle can be moved by the distance equal to the integer part of its velocity (notation $\lfloor v \rfloor$) not interacting with other particles in the configuration.

The dynamics is defined as follows. First we modify the velocities adding to each of them the corresponding acceleration and observing the restriction that velocities cannot exceed V_{\max} . In order to satisfy the admissibility condition we compare each of the resulting velocities to the distance to the next particle to the right (denote it by ℓ) and take the minimum if needed. Thus the modified velocity can be written as follows: $\min\{v + a, V_{\max}, \ell\}$. After that each particle is moved to the right by the distance equal to the integer part of its velocity $\lfloor v \rfloor$ (see Fig. 2). It is immediate to check that this model satisfies all properties assumed in Section 2.

Under the restriction $V_{\max} = 1$ and all accelerations are identical ergodic properties of this model have been studied in [2].¹ In the limiting case when the acceleration is equal to one this model coincides exactly with the well-known NS model, while for fractional values of the acceleration it imitates some more complicated non Markov traffic models (see a discussion in

¹In [2] the movement of particles and their acceleration were applied in the opposite order but this makes no difference for statistical properties of the system.

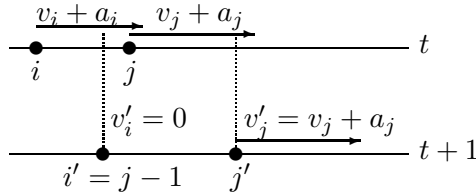


Figure 2: Dynamics of particles acceleration: $i < j$ and $i' < j'$ – the positions of neighboring particles and v_i, v_j, v'_i, v'_j – the corresponding velocities at time t and $t + 1$.

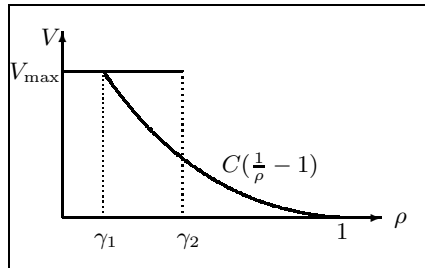


Figure 3: Dependence of the limit average velocity V on the density of particles ρ .

[2]). In [2] it has been shown that that the fractional acceleration leads to a very rich hysteresis type phenomena.

Considered from a bit more general point of view this model describes the motion of particles on a lattice under the action of a constant force represented by the acceleration a , while the presence of the finite maximal velocity may be interpreted as a result of viscosity. Therefore if the particles in the configuration have different masses or sizes it is natural to think that under the action of the same force they will get different accelerations. This is exactly the case we consider here. Fig. 3 summarizes the results of [2] describing the dependence between the average velocity and the particle density of a homogeneous system of particles with $V_{\max} = 1$ and $a \leq 1$. Here $\gamma_1 := (1 + \lceil 1/a \rceil V_{\max})^{-1}$, $\gamma_2 := (1 + V_{\max})^{-1}$, and $\lceil \cdot \rceil$ stands for the smallest integer not smaller than the considered number.

The most interesting part of Fig. 3 corresponds to the region of densities between two critical values γ_i where the one to one correspondence between the average velocity and the density breaks down. In fact, the correspondence in this region is even more complicated and we refer the reader for the detailed analysis to [2].

Clearly in the absence of obstacles the particles in the configurations are moving freely under their accelerations until they get the largest velocity. All peculiarities of the traffic are connected to ‘jams’ (when the motion of a particle is blocked by another one due to the admissibility condition) as the only possible obstacles to the free motion of particles.

We shall say that a *jam* J is a locally maximal collection of consecutive particles in a given configuration having velocities strictly smaller than the maximal allowed one V_{\max} .

The number of particles and their positions in a jam may change with time: leading particles are becoming free (i.e. getting the maximal allowed velocity V_{\max}) and some new particles are joining the jam coming from behind. However, only one such change at a time might happen, and, in particular, a jam cannot split into several new jams. Therefore we can analyze how a given jam changes with time and the main quantity of interest for us here is the minimal number of iterations after which the jam will cease to exist. Denote by $J(t)$ the segment corresponding to the given jam at the moment t (in this notation $J(0)$ is the original jam). Then by the

In fact, this estimate is optimal if $V_{\max} = 1$. If $V_{\max} > 1$ several particles may accelerate simultaneously which might diminish the life-time significantly.

Denoting $a(x) := \inf a_i$, where the infimum is taken over accelerations of all particles present in the configuration x , and assuming that $a(x) > 0$ we get that the life-time of a jam J in this configuration satisfies the inequality

$$\tau(J) \leq \lceil V_{\max}/a(x) \rceil \cdot |BA(J)|. \quad (4.2)$$

Here $|BA(J)|$ stands for the number of particles in the basin of attraction of the jam J .

Consider now what is happening on the level of individual particles in a configuration having no infinite life-time jams. Clearly each particle can move through a jam spending there only a finite time, but it might be possible that ahead of a given particle there is an infinite sequence of jams with monotonously growing (albeit finite) life-times. Therefore some additional care is needed to show that this does not prevent the particle to become eventually free (i.e. moving at maximal velocity).

Recall that the average velocity of a particle in a configuration x located initially (at $t = 0$) at the site $i \in \mathbb{Z}$ is defined as

$$V(x, i) := \lim_{t \rightarrow \infty} \frac{1}{t} L(x, i, t)$$

provided that it is well defined. Here $L(x, i, t)$ is the distance covered by this particle during the time t . A simple argument similar to Lemma 2.2 in [2] shows that if the above limit is well defined for a certain particle in the given configuration then it does not depend on the initial coordinate of the particle and is the same for all particles. Choose any pair of consecutive particles located initially at sites $i < j$. Under dynamics the distance between these particles changes according to the difference between their velocities, which might take values between 0 and V_{\max} . Since the left particle can be slowed down only by the right one, we see that for any moment of time t the distance between the particles can be enlarged at most by CV_{\max} , where the constant $C < \infty$ depends on the accelerations of these particles but not on time. Thus

$$0 \leq (j + L(x, j, t)) - (i + L(x, i, t)) \leq j - i + CV_{\max}$$

or

$$j - i \leq L(x, j, t) - L(x, i, t) \leq CV_{\max}.$$

Dividing by t and using the definition of the time average velocity we get

$$|V(x, i, t) - V(x, j, t)| \leq \max \left\{ \frac{CV_{\max}}{t}, \frac{j - i}{t} \right\} \xrightarrow{t \rightarrow \infty} 0. \quad (4.3)$$

Thus $V(x, i) = V(x, j)$. Using the same argument one extends this result to neighboring particles, and repeating it to all particles in the configuration. Therefore we may drop the index i in the definition of the average velocity.

In order to show that the absence of infinite life-time jams leads to the free eventual motion of particles it is enough to prove that $V(x) = V_{\max}$. Consider a partition of the integer lattice by nonintersecting finite BAs corresponding to jams in the configuration x and their complement (gaps between the jams). Choose one of those BAs and denote by i the position of the first particle preceding it. According to the definition of the BA this particle will never join the jam corresponding to the BA. Moreover, since we assume that the BAs included in the partition are disjoint, the particle never joins any jam corresponding to the elements of the partition. Thus it will have no obstacles in its motion and after at most $\lceil V_{\max}/a \rceil$ time steps it will get the maximal velocity V_{\max} .

This together with the independence of the average velocity on the initial position of the particle proves that $V(x) = V_{\max}$.

It remains to discuss the connection between the condition of the absence of infinite life-time jams and the particle density.

In principle, arguments used in the proof of Lemma 3.1 of [2] can be extended to the case of ‘variable’ accelerations but only under the condition $V_{\max} = 1$. Even in this case the calculations are becoming rather messy. Therefore instead of getting more sharp estimates working only in the case of ‘slow’ particles we shall obtain much more rough estimates of the critical density under which there are no infinite life-time jams being valid for any V_{\max} .

Consider an infinite life-time jam $J = x[m, n]$. Recall that according to the definition of a jam a free particle located initially at the site preceding the BA of a jam cannot join the jam. Therefore using the estimate for the life-time of a jam (4.2) one can show that if the segment $x[n - L + 1, n]$ is contained in the BA of this jam then

$$N_v < N_p \lceil V_{\max}/a(x) \rceil.$$

Here N_p is the number of particles in the segment $x[n - L + 1, n]$ and $N_v := L - N_p$ is the number of vacancies in the segment. Therefore

$$\frac{N_p}{L} = \frac{N_p}{N_p + N_v} > (\lceil V_{\max}/a(x) \rceil + 1)^{-1}.$$

Passing to the limit as $L \rightarrow \infty$ and using that the BA of this jam is infinite as well we come to the estimate of the critical density below which there are no infinite life-time jams:

$$\rho(x) \geq \gamma_1 := \frac{1}{\lceil V_{\max}/a(x) \rceil + 1}. \quad (4.4)$$

Interestingly, the upper bound γ_2 of the densities when there exist configurations consisting of only free moving particles does not depend on the accelerations. To calculate γ_2 consider the most ‘compressed’ configuration of free particles. Since each of them is moving at velocity V_{\max} then the distance between neighboring particles cannot be smaller than V_{\max} , which immediately yields $\gamma_2 := (V_{\max} + 1)^{-1}$.

Similarly to the results of [2] corresponding to the case of the constant acceleration and ‘slow’ particles with $V_{\max} = 1$ one expects that the model under considerations has two distinct ergodic (unmixed) phases with two critical values of the particle density. When the density is below the lowest critical value, the steady state of the model corresponds to the “free-flowing” (or “gaseous”) phase. When the density exceeds the second critical value the model produces large, persistent, well-defined traffic jams, which correspond to the “jammed” (or “liquid”) phase. Between the two critical values each of these phases may take place, which can be interpreted as an “overcooled gas” phase when a small perturbation can change drastically gas into liquid.

The estimates we obtained so far correspond to the “gaseous” phase. It can be shown that when the particle density exceeds the second critical value γ_2 not only the jams are unavoidable but infinitely many infinite life-time jams are present. This explains why we call this phase as “jammed”. As we already mentioned high maximal velocity $V_{\max} > 1$ leads to the appearance of dynamic jams completely absent in the case of ‘slow’ particles, which in turn complicates the analysis of the region between the critical values. Clearly the presence of different accelerations leads to even more complicated dependence between the average velocities and the particle densities and as we expect without the knowledge of the distribution of the accelerations one cannot derive this dependence.

Recalling the dependence of the average velocity and the density in the “liquid” phase obtained exactly in the homogeneous case (see Fig 3) we see that it depends heavily on the acceleration. Therefore in the non homogeneous case one cannot expect to get any functional dependence here.

5 Conclusion

In this paper it has been shown that in a number of cases the analysis of non homogeneous lattice interacting particle systems (IPS) may be reduced to the analysis of homogeneous ones. In distinction to known examples of this sort where the inhomogeneity was due to varying hopping rates (so one expects a certain self averaging) we have shown that a rigorous reduction may be achieved even when the sizes of particles are different. Interestingly, in the latter case the details of the size distribution does not play an important role (see the derivation of the limit average velocity and density in Section 2). Additionally we have considered two different situation when the inhomogeneity come in the form of varying particle velocities or accelerations (which can be considered as a version of varying hopping rates). In both cases we obtained either an exact reduction to the homogeneous system or rigorous estimates of important statistical quantities.

The analysis made in the paper was restricted to the discrete time systems, i.e. to IPS with the parallel updating. On the other hand, all results of Sections 2 and 3 hold in complete generality both for continuous time IPS and systems with random sequential updating.

To finalize let us describe a few open problems in the field.

Due to the clear connections of the IPS under study to traffic flow modelling it would be of interest to extend our results about the mapping of the IPS with particles of different size to multilane traffic models, where the particles are moving and exchanging positions along several lattice lines. Multilane traffic models with identical particles were studied mathematically, e.g. in [1], where the exact dependence between limit average velocities and particle densities were obtained. We expect that a version of the size reduction developed in Section 2 should work here but the problem with the non homogeneous case is that it is not clear how to take into account the change of particles of different sizes between the lanes.

Another set of questions is related to random versions of the deterministic IPS discussed in Section 4, when the movement of particles happen with a certain (may be non homogeneous again) probability. At the moment nothing is known rigorously about such systems and it would be of interest to prove the existence and uniqueness of invariant distributions corresponding to each value of the particle density in the true probabilistic setting.

Throughout the paper we have considered only particle configurations having densities, i.e. being spatially ergodic with respect to the standard shift-map. Applying the ideas developed in [2, 1] one can extend our results to a more general setting using lower and upper densities instead of the usual density. From this point of view it is of interest to study limit statistics corresponding to particle configurations having different *left* and *right* densities: $\rho_{\text{left}}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} N_p(x[-n, -1]) = \alpha$, $\rho_{\text{right}}(x) := \lim_{n \rightarrow \infty} \frac{1}{n} N_p(x[1, n]) = \beta$. One can think about this as an imitation of ‘open’ systems with the entrance rate α and the exit rate β .

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