A family of the Poisson brackets compatible with the Sklyanin bracket

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We introduce a family of compatible Poisson brackets on the space of 2×2 polynomial matrices, which contains the Sklyanin bracket, and use it to derive a multi-Hamiltonian structure for a set of integrable systems that includes XXX Heisenberg magnet, the open and periodic Toda lattices, the discrete self-trapping model and the Goryachev-Chaplygin gyrostat.

1 Introduction.

The ingenious discovery of Magri [6, 7] that integrable Hamiltonian systems usually prove to be bi-Hamiltonian, and vice versa, leads us to the following fundamental problem: given a dynamical system which is Hamiltonian with respect to a Poisson bracket $\{.,.\}_0$, how to find another Poisson bracket $\{.,.\}_1$ compatible with initial bracket and such that our system is Hamiltonian with respect to both brackets. This, along with the related problem of classification of compatible Poisson structures, is nowadays a subject of intense research, see e.g. [6, 7, 2, 14] and references therein.

In this paper we study a class of finite-dimensional Liouville integrable systems described by the representations of the quadratic r-matrix Poisson algebra, or the Sklyanin algebra:

$$\{\overset{1}{T}(\lambda),\overset{2}{T}(\mu)\}=[r(\lambda-\mu),\overset{1}{T}(\lambda)\overset{2}{T}(\mu)\,]\,,\tag{1.1}$$

Here $\overset{1}{T}(\lambda)=T(\lambda)\otimes \mathrm{I}\,,\ \overset{2}{T}(\mu)=\mathrm{I}\otimes T(\mu)$ and $r(\lambda-\mu)$ is a classical r-matrix [8]-[11].

The main result of the present paper is a family of the Poisson brackets $\{.,.\}_k$, which is compatible with the Sklyanin bracket (1.1), in the simplest case of the 4×4 rational r-matrix

$$r(\lambda-\mu)=\frac{\eta}{\lambda-\mu}\Pi,\qquad \Pi=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\,,\quad \eta\in\mathbb{C}\,,\tag{1.2}$$

and 2×2 matrix $T(\lambda)$, which depends polynomially on the parameter λ

$$\begin{aligned}T(\lambda)&=\begin{pmatrix}A(\lambda)&B(\lambda)\\C(\lambda)&D(\lambda)\end{pmatrix}\\&=\begin{pmatrix}\alpha\lambda^n+A_1\lambda^{n-1}+\ldots+A_n&\beta\lambda^n+B_1\lambda^{n-1}+\ldots+B_n\\\gamma\lambda^n+C_1\lambda^{n-1}+\ldots+C_n&\delta\lambda^n+D_1\lambda^{n-1}+\ldots+D_n\end{pmatrix}.\end{aligned}\tag{1.3}$$

The leading coefficients $\alpha, \beta, \gamma, \delta$ and $2n$ coefficients of the $\det T(\lambda)$

$$d(\lambda) = \det T(\lambda) = (\alpha\delta - \beta\gamma)\lambda^{2n} + Q_1\lambda^{2n-1} + \cdots + Q_{2n}\,. \qquad (1.4)$$

are Casimirs of the bracket (1.1). Therefore, we have a $4n$-dimensional space of the coefficients A_i, B_i, C_i and D_i with $2n$ Casimir operators Q_i, leaving us with n degrees of freedom.

For so-called open lattices independent Poisson involutive integrals of motion $H_i^o = A_i$, $i = 1, \ldots, n$, are given by the coefficients of the entry $A(\lambda)$:

$$A(\lambda) = \alpha\lambda^n + H_1^o\lambda^{n-1} + \cdots H_n^o, \qquad \{H_i^o, H_j^o\} = 0\,. \qquad (1.5)$$

In generic case integrals of motion are given by the coefficients of the $\mathrm{tr}T(\lambda)$:

$$\mathrm{tr}T(\lambda) = (\alpha + \delta)\lambda^n + H_1\lambda^{n-1} + \cdots H_n, \qquad \{H_i, H_j\} = 0\,. \qquad (1.6)$$

These integrals of motion define two Liouville integrable systems, which are our generic models for the whole paper. Bi-hamiltonian description of these models gives rise to the bi-hamiltonian description of the Goryachev-Chaplygin gyrostat [8], open and periodic Toda lattice [9], inhomogeneous Heisenberg magnet [11] and the discrete self-trapping (DST) model [5].

2 The compatible bracket

In this section, we describe the Poisson bracket compatible with the Sklyanin bracket. The Poisson brackets $\{.,.\}_0$ and $\{.,.\}_1$ are compatible if every linear combination of them is still a Poisson bracket. The corresponding compatible Poisson tensors P_0 and P_1 satisfy to the following equations

$$[\![P_0, P_0]\!] = [\![P_0, P_1]\!] = [\![P_1, P_1]\!] = 0, \qquad (2.1)$$

where $[\![.,.]\!]$ is the Schouten bracket [2, 6, 7]. Remind that on a smooth finite-dimensional manifold $\mathscr{M}$ the Schouten bracket of two bivectors X and Y is an antisymmetric contravariant tensor of rank three and its components in local coordinates z_m read

$$[\![X, Y]\!]^{ijk} = -\sum_{m=1}^{dim\,\mathscr{M}} \left(X^{mk}\frac{\partial Y^{ij}}{\partial z_m} + Y^{mk}\frac{\partial X^{ij}}{\partial z_m} + \mathrm{cycle}(i,j,k) \right).$$

2.1 Open lattices

The Sklyanin bracket (1.1) amounts to having the following Poisson brackets between the entries $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ of the matrix $T(\lambda)$:

$$\begin{aligned}
&\{A(\lambda),A(\mu)\}_0=\{B(\lambda),B(\mu)\}_0=\{C(\lambda),C(\mu)\}_0=\{D(\lambda),D(\mu)\}_0=0,\\
&\{B(\lambda),A(\mu)\}_0=\frac{\eta}{\lambda-\mu}\Big(B(\lambda)A(\mu)-B(\mu)A(\lambda)\Big),\\
&\{C(\lambda),A(\mu)\}_0=\frac{-\eta}{\lambda-\mu}\Big(C(\lambda)A(\mu)-C(\mu)A(\lambda)\Big),\\
&\{B(\lambda),C(\mu)\}_0=\frac{\eta}{\lambda-\mu}\Big(D(\lambda)A(\mu)-D(\mu)A(\lambda)\Big).\\
&\{B(\lambda),D(\mu)\}_0=\frac{-\eta}{\lambda-\mu}\Big(B(\lambda)D(\mu)-B(\mu)D(\lambda)\Big),\\
&\{C(\lambda),D(\mu)\}_0=\frac{\eta}{\lambda-\mu}\Big(C(\lambda)D(\mu)-C(\mu)D(\lambda)\Big),\\
&\{A(\lambda),D(\mu)\}_0=\frac{\eta}{\lambda-\mu}\Big(C(\lambda)B(\mu)-C(\mu)B(\lambda)\Big),
\end{aligned}\tag{2.2}$$

In (1.1) matrix $r(\lambda-\mu)$ satisfies the Yang-Baxter equation, which ensures the Jacobi identity for the brackets (2.2).

Proposition 1 *The Sklyanin bracket (1.1),(2.2) is compatible with the following bracket* $\{.,.\}_1$*:*

$$\begin{aligned}
&\{A(\lambda),A(\mu)\}_1=\{B(\lambda),B(\mu)\}_1=\{C(\lambda),C(\mu)\}_1=0,\\
&\{B(\lambda),A(\mu)\}_1=\frac{\eta}{\lambda-\mu}\Big(\lambda B(\lambda)A(\mu)-\mu B(\mu)A(\lambda)\Big)-\frac{\eta\,\beta}{\alpha}A(\lambda)A(\mu),\\
&\{C(\lambda),A(\mu)\}_1=\frac{-\eta}{\lambda-\mu}\Big(\lambda C(\lambda)A(\mu)-\mu C(\mu)A(\lambda)\Big)+\frac{\eta\,\gamma}{\alpha}A(\lambda)A(\mu),\\
&\{B(\lambda),C(\mu)\}_1=\frac{\eta}{\lambda-\mu}\Big(\lambda D(\lambda)A(\mu)-\mu D(\mu)A(\lambda)\Big)-\frac{\eta\,\delta}{\alpha}A(\lambda)A(\mu),\\
&\{B(\lambda),D(\mu)\}_1=\frac{-\eta\,\lambda}{\lambda-\mu}\Big(B(\lambda)D(\mu)-B(\mu)D(\lambda)\Big)+\eta A(\lambda)\left(\frac{\beta}{\alpha}D(\mu)-\frac{\delta}{\alpha}B(\mu)\right),\\
&\{C(\lambda),D(\mu)\}_1=\frac{\eta\,\lambda}{\lambda-\mu}\Big(C(\lambda)D(\mu)-C(\mu)D(\lambda)\Big)-\eta A(\lambda)\left(\frac{\gamma}{\alpha}D(\mu)-\frac{\delta}{\alpha}C(\mu)\right),\\
&\{A(\lambda),D(\mu)\}_1=\frac{\eta\,\lambda}{\lambda-\mu}\Big(C(\lambda)B(\mu)-C(\mu)B(\lambda)\Big)-\eta A(\lambda)\left(\frac{\gamma}{\alpha}B(\mu)-\frac{\beta}{\alpha}C(\mu)\right),
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\{D(\lambda),D(\mu)\}_1 &= \frac{\eta\,\gamma}{\alpha}\Big(D(\lambda)B(\mu)-D(\mu)B(\lambda)\Big)-\frac{\eta\,\beta}{\alpha}\Big(D(\lambda)C(\mu)-D(\mu)C(\lambda)\Big)\\ &+ \frac{\eta\,\delta}{\alpha}\Big(B(\lambda)C(\mu)-B(\mu)C(\lambda)\Big)\,. \end{aligned} \tag{2.4}$$

Proof: It is sufficient to check the statement on an open dense subset of the Sklyanin algebra defined by the assumption that $A(\lambda)$ and $B(\lambda)$ are co-prime and all roots of $A(\lambda)$ are distinct.

This assumption allows us to construct a separation representation for the Sklyanin algebra (1.1). In this special representation one has n pairs of Darboux variables, λ_i, μ_i, $i=1,\dots,n$, having the standard Poisson brackets,

$$\{\lambda_i,\lambda_j\}_0=\{\mu_i,\mu_j\}_0=0,\qquad \{\lambda_i,\mu_j\}_0=\delta_{ij}, \tag{2.5}$$

with the λ-variables being n zeros of the polynomial $A(\lambda)$ and the μ-variables being values of the polynomial $B(\lambda)$ at those zeros,

$$A(\lambda_i)=0,\qquad \mu_i=\eta^{-1}\ln B(\lambda_i),\qquad i=1,\dots,n. \tag{2.6}$$

The interpolation data (2.6) plus n identities

$$B(\lambda_i)C(\lambda_i)=-d(\lambda_i)$$

allow us to construct the needed separation representation for the whole algebra:

$$\begin{aligned} &A(\lambda)=\alpha(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n),\\ &B(\lambda)=A(\lambda)\left(\frac{\beta}{\alpha}+\sum_{i=1}^n\frac{\mathrm{e}^{\eta\mu_i}}{(\lambda-\lambda_i)A'(\lambda_i)}\right),\\ &C(\lambda)=A(\lambda)\left(\frac{\gamma}{\alpha}-\sum_{i=1}^n\frac{d(\lambda_i)\ \mathrm{e}^{-\eta\mu_i}}{(\lambda-\lambda_i)A'(\lambda_i)}\right),\\ &D(\lambda)=\frac{d(\lambda)+B(\lambda)C(\lambda)}{A(\lambda)}\,. \end{aligned} \tag{2.7}$$

The coefficients of the determinant $d(\lambda)$ (1.4) are Casimir elements for the both brackets $\{.,.\}_0$ and $\{.,.\}_1$ and, therefore, we can easy calculate the bracket $\{.,.\}_1$ (2.3)–(2.4) in (λ,μ)-variables

$$\{\lambda_i,\lambda_j\}_1=\{\mu_i,\mu_j\}_1=0,\qquad \{\lambda_i,\mu_j\}_1=\lambda_i\delta_{ij}, \tag{2.8}$$

In order to complete the proof we have to check that brackets (2.8) is compatible with the canonical brackets (2.5). The compatibility of the brackets (2.5),(2.8) implies the compatibility of the brackets (2.2),(2.3) and vice versa.

The (λ,μ)-variables (2.6) are so-called special Darboux-Nijenhuis coordinates [6, 7, 2] because

$$P_0=\begin{pmatrix}0 & \mathrm{I}\\ -\mathrm{I} & 0\end{pmatrix},\qquad P_1=\begin{pmatrix}0 & \mathrm{diag}(\lambda_1,\dots,\lambda_n)\\ -\mathrm{diag}(\lambda_1,\dots,\lambda_n) & 0\end{pmatrix},$$

and the corresponding recursion operator N takes the diagonal form

$$N = P_1 P_0^{-1} = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \mu_i} \otimes d\mu_i \right). \tag{2.9}$$

These Poisson tensors P_0 and P_1 satisfy to the equations (2.1) and the Nijenhuis torsion of N vanishes as a consequence of the compatibility between P_0 and P_1.

Proposition 2 *Brackets (2.5) and (2.8) between (λ, μ)-variables belong to a whole family of compatible Poisson brackets $\{.,.\}_k$ associated with the Poisson tensors*

$$P_k = N^k P_0 = \begin{pmatrix} 0 & \mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k) \\ -\mathrm{diag}(\lambda_1^k, \dots, \lambda_n^k) & 0 \end{pmatrix}, \qquad k = 0, \dots, n.$$

In the matrix form, these brackets are equal to

$$\begin{aligned} \left\{ \overset{1}{T}(\lambda), \overset{2}{T}(\mu) \right\}_k &= r_{12}^{[k]}(\lambda,\mu) \overset{1}{T}(\lambda) \overset{2}{T}(\mu) - \overset{1}{T}(\lambda) \overset{2}{T}(\mu)\, r_{21}^{[k]}(\lambda,\mu) \\ &+ \overset{1}{T}(\lambda)\, s_{12}^{[k]}(\lambda,\mu) \overset{2}{T}(\mu) - \overset{2}{T}(\mu)\, s_{21}^{[k]}(\lambda,\mu) \overset{1}{T}(\lambda) . \end{aligned} \tag{2.10}$$

Here

$$r_{12}^{[k]}(\lambda,\mu) = \frac{\eta}{\lambda-\mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{\lambda^k+\mu^k}{2} & \mu^k & 0 \\ 0 & \lambda^k & 1-\frac{\lambda^k+\mu^k}{2} & 0 \\ 0 & \rho_C^{[k]} & -\rho_C^{[k]} & 1 \end{pmatrix}, \qquad r_{21}^{[k]}(\lambda,\mu) = \frac{\eta}{\lambda-\mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\frac{\lambda^k+\mu^k}{2} & \lambda^k & \rho_B^{[k]} \\ 0 & \mu^k & 1-\frac{\lambda^k+\mu^k}{2} & -\rho_B^{[k]} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_{12}^{[k]}(\lambda,\mu) = \frac{\eta}{\lambda-\mu} \begin{pmatrix} 0 & \rho_B^{[k]} & 0 & 0 \\ 0 & \frac{\lambda^k-\mu^k}{2} & 0 & 0 \\ \rho_C^{[k]} & \rho_D^{[k]} & \frac{\lambda^k-\mu^k}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad s_{21}^{[k]}(\lambda,\mu) = \Pi s_{12}^{[k]}(\lambda,\mu) \Pi. \tag{2.11}$$

and

$$\rho_X^{[k]} = \frac{\lambda^k X(\lambda)}{A(\lambda)} - \frac{\mu^k X(\mu)}{A(\mu)}, \qquad \textit{where} \qquad X = B, C, D,$$

is a difference of two polynomials, which are quotients of polynomials in variables λ and μ over a field.

Proof: At $k = 0$ one has $\rho_B^{[0]} = 0$, $\rho_C^{[0]} = 0$ and $\rho_D^{[0]} = 0$, so the bracket (2.10) coincides with the Sklyanin bracket (1.1). At $k = 1$ we have

$$\rho_B^{[1]} = \frac{\beta(\lambda-\mu)}{\alpha}, \qquad \rho_C^{[1]} = \frac{\gamma(\lambda-\mu)}{\alpha}, \qquad \rho_D^{[1]} = \frac{\delta(\lambda-\mu)}{\alpha}$$

and bracket (2.10) coincides with the bracket (2.3).

At $k > 1$ one can easily check that k-th brackets (2.10) between polynomials $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ (2.7) imply the brackets

$$\{\lambda_i, \lambda_j\}_k = \{\mu_i, \mu_j\}_k = 0, \qquad \{\lambda_i, \mu_j\}_k = \lambda_i^k \delta_{ij}. \tag{2.12}$$

and vice versa. This completes the proof.

To proceed further we need to recall that the normalized traces of the powers of N

$$J_m = \frac{1}{2m}\,\mathrm{trace}\,N^m = \sum_{i=1}^{n} \lambda_i^m, \qquad m = 1, \ldots, n. \tag{2.13}$$

are integrals of motion satisfying Lenard-Magri recurrent relations [6, 7]

$$P_0 dJ_1 = 0, \qquad X_{J_i} = P_0 dJ_i = P_1 dJ_{i-1}, \qquad P_1 dJ_n = 0. \tag{2.14}$$

By definition (2.7) polynomial

$$A(\lambda) = \alpha\lambda^n + A_1\lambda^{n-1} + \ldots + A_n = \alpha\prod_{i=1}^{n}(\lambda - \lambda_i)$$

is directly proportional to the minimal characteristic polynomial of N (2.9)

$$\Delta_N(\lambda) = \left(\det(N - \lambda \mathrm{I})\right)^{1/2} = \prod_{i=1}^{n}(\lambda - \lambda_i).$$

Since Hamiltonians H_i^o (1.5) are related with integrals of motion J_m (2.13) by the triangular Newton formulas

$$\alpha J_1 = H_1^o, \quad \alpha J_2 = H_2^o + \frac{(H_1^o)^2}{2}, \quad \alpha J_3 = H_3^o + H_2^o H_1^o + \frac{(H_1^o)^3}{3}, \ldots.$$

As a consequence of the recursion relations (2.14), the Hamiltonians H_i^o, $i = 1, \ldots, n$, satisfy the Fröbenius recursion relations

$$N^*\, dH_i^o = dH_{i+1}^o - \alpha^{-1} A_i\, dH_1^o, , \tag{2.15}$$

where $N^* = P_0^{-1} P_1$ and $H_{n+1}^o = 0$. Such as $A_i = H_i^o$ a straightforward computation shows that they are equivalent

$$N^*\, dA(\lambda) = \lambda\, dA(\lambda) + A(\lambda) dA_1\,.$$

The special Darboux-Nijenhuis coordinates λ_i, μ_i are variables of separation of the action-angle type [2], i.e. the corresponding separated equations are trivial

$$\{H_i^o, \lambda_j\} = \{J_i, \lambda_j\} = 0\,, \qquad i, j = 1, \ldots, n.$$

We can introduce another separated coordinates u_i, v_i, which are the so-called Sklyanin variables defined by

$$B(u_i) = 0, \qquad v_i = -\eta^{-1} \ln A(u_i)\,, \qquad i = 1, \ldots, n.$$

The separation representation of the algebra in (u,v)-variables has the form

$$\begin{aligned}
B(\lambda) &= \beta(\lambda-u_1)(\lambda-u_2)\cdots(\lambda-u_n),\\
A(\lambda) &= B(\lambda)\left(\frac{\alpha}{\beta}+\sum_{i=1}^{n}\frac{\mathrm{e}^{-\eta v_i}}{(\lambda-u_i)B'(u_i)}\right),\\
D(\lambda) &= B(\lambda)\left(\frac{\delta}{\beta}+\sum_{i=1}^{n}\frac{d(u_i)\,\mathrm{e}^{\eta v_i}}{(\lambda-u_i)B'(u_i)}\right),\\
C(\lambda) &= \frac{A(\lambda)D(\lambda)-d(\lambda)}{B(\lambda)}.
\end{aligned}$$

Substituting matrix $T(\lambda)$ (1.3) with these entries into the brackets $\{.,.\}_k$ (2.10) at $k=0,1$ one gets that u_i, v_j coordinates are Darboux variables with respect to the Sklyanin bracket

$$\{u_i,u_j\}_0=\{v_i,v_j\}_0=0,\qquad \{u_i,v_j\}_0=\delta_{ij}, \tag{2.16}$$

whereas the second brackets look like

$$\{u_i,u_j\}_1=0,\qquad \{u_i,v_j\}_1=u_i\delta_{ij}-\frac{\beta\,A(u_j)}{\alpha\,B'(u_j)},\qquad \{v_i,v_j\}=\frac{A'(u_i)}{B'(u_i)}-\frac{A'(u_j)}{B'(u_j)}.$$

The corresponding separated equations

$$\{A(\lambda),u_j\}_k=\lambda^k\,A(u_j)\prod_{i\neq j}^{n-1}\frac{\lambda-u_i}{u_j-u_i},\qquad j=1,\ldots,n. \tag{2.17}$$

are linearized by the Abel transformation on the algebraic curve defined by $\mathrm{e}^{-\eta v_i}=A(u_i)$, see [9, 3, 12] and references within.

The special Darboux-Nijenhuis coordinates are dual to the Sklyanin variables. Namely, λ_i,μ_i are roots of polynomial $A(\lambda)$ and values of polynomial $B(\lambda)$ at $\lambda=\lambda_i$, while u_i,v_i are roots of polynomial $B(\lambda)$ and values of polynomial $A(\lambda)$ at $\lambda=u_i$.

2.2 Generic model

There are many other Poisson brackets compatible with the standard one (2.5). The main property of the proposed above bracket $\{.,.\}_1$ (2.3)–(2.4) is that

$$\{A(\lambda),A(\mu)\}_0=\{A(\lambda),A(\mu)\}_1=0\,.$$

It ensures that integrals of motion H_i^o for the open lattices are in bi-involution

$$\left\{H_i^o,H_j^o\right\}_0=\left\{H_i^o,H_j^o\right\}_1=0.$$

In this subsection we are looking for bracket $\{.,.\}_1'$, which has to guarantee the similar property for generic integrals of motion H_i (1.6) from $\mathrm{tr}T(\lambda)$

$$\{H_i,H_j\}_0=\{H_i,H_j\}_1'=0,\qquad i,j=1,\ldots,n.$$

Remind that $\{.,.\}_0$ is the Sklyanin bracket (1.1), which already has the necessary property

$$\{\mathrm{tr}T(\lambda),\mathrm{tr}T(\mu)\}_0=0\,.$$

The following propositions can be ascertained by means of direct calculations.

Proposition 3 *If $\alpha = \delta$ and $\beta = \gamma = 0$ in $T(\lambda)$ (1.3), then*

$$\{\mathrm{tr}T(\lambda), \mathrm{tr}T(\mu)\}_1 = 0\,. \tag{2.18}$$

So, the desired bracket $\{.,.\}'_1$ may be obtained from the bracket $\{.,.\}_1$ (2.3)–(2.4) by using special canonical transformations, which are generated by the suitable transformations of the matrix $T(\lambda)$.

Proposition 4 *The Sklyanin bracket (1.1) is invariant with respect to transformation*

$$T(\lambda) \to V_1\, T(\lambda)\, V_2, \qquad V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}, \qquad \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}\,, \tag{2.19}$$

where $V_{1,2}$ are numerical matrices. If

$$\beta_1\gamma_2 + \delta_1\delta_2 = 0,$$

the bracket (2.3)–(2.4) after transformation (2.19) has the necessary property

$$\{\mathrm{tr}T(\lambda), \mathrm{tr}T(\mu)\}'_1 = 0\,. \tag{2.20}$$

We present an explicit form of the bracket $\{.,.\}'_1$ in the Section 4 devoted to the periodic Toda lattice.

3 The Heisenberg magnet

Another important representation of the quadratic algebra with the generators A_i, B_i, C_i and D_i comes as a consequence of the co-multiplication property of the Sklyanin algebra (1.1). Essentially, it means that the matrix $T(\lambda)$ (1.3) can be factorized into a product of elementary matrices, each containing only one degree of freedom. In this picture, our main model turns out to be an n-site Heisenberg magnet, which is an integrable lattice of n sl(2) spins with nearest neighbor interaction.

In the lattice representation the matrix $T(\lambda)$ (1.3) acquires the following form:

$$T(\lambda) = L_1(\lambda - c_1)\; L_2(\lambda - c_2)\; \cdots\; L_n(\lambda - c_n)\;, \tag{3.1}$$

with

$$L_m(\lambda) = \begin{pmatrix} \lambda - s_3^{(m)} & s_1^{(m)} + \mathrm{i}s_2^{(m)} \\ s_1^{(m)} - \mathrm{i}s_2^{(m)} & \lambda + s_3^{(m)} \end{pmatrix}, \qquad m = 1, \ldots, n. \tag{3.2}$$

Here $s_3^{(m)}$ are dynamical variables, c_m are arbitrary numbers and $\mathrm{i} = \sqrt{-1}\,$.

Substituting matrix (3.1) into the Sklyanin bracket (1.1) and brackets (2.3)-(2.4) at $\eta = \mathrm{i}$ one gets canonical brackets on the direct sum of sl(2)

$$\left\{ s_i^{(m)}, s_j^{(m)} \right\}_0 = \varepsilon_{ijk}\, s_k^{(m)}\,, \tag{3.3}$$

and second compatible brackets

$$\left\{s_i^{(m)},s_j^{(m)}\right\}_1=\varepsilon_{ijk}\,s_k^{(m)}\big(c_m-s_3^{(m)}\big),\qquad \left\{s_i^{(m)},s_j^{(\ell)}\right\}_1=\left(P_1^{(m\ell)}\right)_{ij},\quad m\neq\ell,$$

where ε_{ijk} is the totally skew-symmetric tensor and

$$P_1^{(m\ell)}=\begin{pmatrix} -\mathrm{i}\big(s_3^{(m)}s_3^{(\ell)}+s_2^{(m)}s_2^{(\ell)}\big) & \mathrm{i}s_2^{(m)}s_1^{(\ell)}-s_3^{(m)}s_3^{(\ell)} & \mathrm{i}s_3^{(m)}\big(s_1^{(\ell)}-\mathrm{i}s_2^{(\ell)}\big)\\ \mathrm{i}s_1^{(m)}s_2^{(\ell)}+s_3^{(m)}s_3^{(\ell)} & -\mathrm{i}\big(s_3^{(m)}s_3^{(\ell)}+s_1^{(m)}s_1^{(\ell)}\big) & -s_3^{(m)}\big(s_1^{(\ell)}-\mathrm{i}s_2^{(\ell)}\big)\\ -\mathrm{i}\big(s_1^{(m)}+\mathrm{i}s_2^{(m)}\big)s_3^{(\ell)} & \big(s_1^{(m)}+\mathrm{i}s_2^{(m)}\big)s_3^{(\ell)} & -\mathrm{i}\big(s_1^{(m)}+\mathrm{i}s_2^{(m)}\big)\big(s_1^{(\ell)}-\mathrm{i}s_2^{(\ell)}\big)\end{pmatrix}.$$

The corresponding Poisson tensors P_0 and P_1 are degenerate and, therefore, the Hamiltonians H_i^o satisfy the Fröbenius recurrence relations (2.15) in the following form

$$P_1dH_i^o=P_0\left(dH_{i+1}^o-A_idH_1^o\right),\qquad i=1,\ldots,n,\tag{3.4}$$

where $H_{n+1}^o=0$ and $A_i=H_i^o$ are coefficients of the polynomial $A(\lambda)$. The first integrals of motion are

$$H_1^o=\sum_{m=1}^n(c_m-s_3^{(m)}),\qquad H_2^o=\sum_{m>\ell}\big(s_1^{(m)}-\mathrm{i}s_2^{(m)}\big)\big(s_1^{(\ell)}+\mathrm{i}s_2^{(\ell)}\big)-\frac12\sum_{m=1}^n(c_m-s_3^{(m)})^2+\frac{(H_1^o)^2}{2}.$$

Such as $\alpha=\delta$ and $\beta=\gamma=0$ we can use these brackets for the open and periodic lattices simultaneously. It means that Hamiltonians H_i (1.6) from the $\mathrm{tr}T(\lambda)$ satisfy the Fröbenius equations (3.4) too.

4 The Toda lattices

The Toda lattices appear as a specialization of our basic model when the parameters are fixed as follows:

$$\beta=\gamma=\delta=0\qquad\text{and}\qquad\det T(\lambda)=1.\tag{4.1}$$

We also put $\alpha=1$ and $\eta=-1$. In the lattice representation, the monodromy matrix T (1.3) acquires the form

$$T(\lambda)=L_1(\lambda)\cdots L_{n-1}(\lambda)\,L_n(\lambda),\qquad L_i=\begin{pmatrix}\lambda-p_i & -\mathrm{e}^{q_i}\\ \mathrm{e}^{-q_i} & 0\end{pmatrix}.\tag{4.2}$$

Here p_i,q_i are dynamical variables.

4.1 Open lattice

Substituting matrix $T(\lambda)$ (4.2) into the brackets $\{.,.\}_k$ (2.10) at $k=0,1$ one gets that the Poisson tensors P_0 and P_1 in (p,q) variables take the form

$$P_0=\sum_{i=1}^n\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i},\tag{4.3}$$

$$P_1=\sum_{i=1}^{n-1}\mathrm{e}^{q_i-q_{i+1}}\frac{\partial}{\partial p_{i+1}}\wedge\frac{\partial}{\partial p_i}+\sum_{i=1}^n p_i\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}+\sum_{i<j}\frac{\partial}{\partial q_j}\wedge\frac{\partial}{\partial q_i}.$$

Namely this bi-hamiltonian structure of the open Toda lattice was obtained in [1].

For the open Toda lattice the Hamiltonians H_i^o from the $A(\lambda) = \lambda^n + H_1^o\lambda^{n-1} + \ldots + H_n^o$ satisfy the Fröbenius relations (2.15). The first integrals of motion are equal to

$$H_1^o = -\sum_{i=1}^n p_i, \qquad H_2^o = \frac{1}{2}\sum_{i=1}^n {p_i}^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} - \frac{1}{2}\left(\sum_{i=1}^n p_i\right)^2. \tag{4.4}$$

The Sklyanin variables u_i, v_i are introduced as before:

$$B(u_i) = 0, \qquad v_i = -\eta^{-1} \ln A(u_i)\,, \qquad i = 1, \ldots, n-1, \tag{4.5}$$

the only difference now is that this gives only $n-1$ instead of n separation pairs. The missing pair of canonical variables is defined as follows:

$$v_n = \ln\, b_1 = -q_n, \qquad u_n = -a_1 = \sum_{i=1}^n p_i. \tag{4.6}$$

The separation representation of the algebra in (u, v)-variables may be found in [9, 13]. It is easy to prove [13] that (u, v)-variables are Darboux variables

$$\omega = P_0^{-1} = \sum_{i=1}^n du_i \wedge dv_i,$$

and the only nonzero second Poisson brackets are

$$\{u_j, v_i\}_1 = u_i\, \delta_{ij}, \qquad \{u_n, u_i\}_1 = -\mathrm{e}^{-v_n} \tfrac{A(u_i)}{B'(u_i)}, \qquad \{u_n, v_i\}_1 = -\mathrm{e}^{-v_n} \tfrac{A'(u_i)}{B'(u_i)}\,,$$

$$\{v_n, v_i\}_1 = -1, \qquad \{u_n, v_n\}_1 = -\sum_{i=1}^n u_i.$$

Remark 1 From the factorization (4.2) of the monodromy matrix $T(\lambda)$ one gets

$$B_n(\lambda) = -\mathrm{e}^{q_n} A_{n-1}(\lambda) \qquad \Rightarrow \qquad B_n(u_j) = -\mathrm{e}^{q_n} A_{n-1}(\lambda_j) = 0.$$

This implies that for the $(n-1)$-particle chain special Darboux-Nijenhuis variables λ_j coincide with the Sklyanin variables u_j, $i = 1, \ldots, n-1$ for the n-particle chain.

4.2 Periodic lattice

For the Toda lattice $\alpha \neq \delta$ and, therefore, in order to get new bracket $\{.,.\}_1'$ with the the necessary property (2.20) we have to apply transformation (2.19) to the initial bracket $\{.,.\}_1$ (2.3)-(2.4). If we put

$$V_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad V_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

then one gets the following brackets between the entries of $T(\lambda)$:

$$
\begin{aligned}
&\{A(\lambda),A(\mu)\}_1' = \eta\big(B(\lambda)C(\mu)-B(\mu)C(\lambda)\big), \qquad \{D(\lambda),D(\mu)\}_1' = 0,\\
&\{A(\lambda),D(\mu)\}_1' = \frac{\eta\lambda}{\lambda-\mu}\big(C(\lambda)B(\mu)-C(\mu)B(\lambda)\big)\\
&\{B(\lambda),B(\mu)\}_1' = \eta\big(B(\lambda)D(\mu)-B(\mu)D(\lambda)\big),\\
&\{C(\lambda),C(\mu)\}_1' = \eta\big(C(\lambda)D(\mu)-C(\mu)D(\lambda)\big),\\
&\{D(\lambda),B(\mu)\}_1' = \frac{\eta\mu}{\lambda-\mu}\big(B(\lambda)D(\mu)-B(\mu)D(\lambda)\big)\\
&\{D(\lambda),C(\mu)\}_1' = \frac{-\eta\mu}{\lambda-\mu}\big(C(\lambda)D(\mu)-C(\mu)D(\lambda)\big)
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&\{A(\lambda),B(\mu)\}_1' = \tfrac{-\eta\left(\lambda A(\mu)B(\lambda)-\mu A(\lambda)B(\mu)\right)}{\lambda-\mu} + \eta\Big(B(\lambda)D(\mu)+\big(B(\lambda)-C(\lambda)\big)B(\mu)\Big),\\
&\{A(\lambda),C(\mu)\}_1' = \tfrac{\eta\left(\lambda A(\mu)C(\lambda)-\mu A(\lambda)C(\mu)\right)}{\lambda-\mu} - \eta\Big(C(\lambda)D(\mu)+\big(B(\lambda)-C(\lambda)\big)C(\mu)\Big),\\
&\{B(\lambda),C(\mu)\}_1' = \tfrac{\eta\left(\lambda A(\mu)D(\lambda)-\mu A(\lambda)D(\mu)\right)}{\lambda-\mu} - \eta\big(B(\lambda)D(\mu)-D(\lambda)C(\mu)+D(\lambda)D(\mu)\big).
\end{aligned}
\tag{4.8}
$$

According to proposition 4 these brackets have the necessary property (2.20).

Substituting matrix $T(\lambda)$ (4.2) into the brackets (4.7)-(4.8) one gets that the Poisson tensor P_1' in (p,q) variables takes the form

$$
\begin{aligned}
P_1' &= \sum_{i=1}^{n-1} e^{q_i-q_{i+1}}\frac{\partial}{\partial p_{i+1}}\wedge\frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i} + \sum_{i<j}\frac{\partial}{\partial q_j}\wedge\frac{\partial}{\partial q_i}\\
&- \sum_{i=1}^{n}\left(e^{q_1}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_1} + e^{q_n}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_n}\right).
\end{aligned}
$$

For the periodic Toda lattice the Hamiltonians H_1 and H_2 from the $\mathrm{tr}T(\lambda) = \lambda^n + H_1\lambda^{n-1} + \ldots + H_0$ are equal to

$$
H_1 = H_1^o = -\sum_{i=1}^{n} p_i, \qquad H_2 = H_2^o + e^{q_n-q_1} = \frac{1}{2}\sum_{i=1}^{n} {p_i}^2 + \sum_{i=1}^{n} e^{q_i-q_{i+1}} - \frac{1}{2}\left(\sum_{i=1}^{n} p_i\right)^2, \tag{4.9}
$$

where $q_{n+i} = q_i$. These Hamiltonians H_i, $i = 1,\ldots,n$, form the Fröbenius chain

$$
N^* dH_i = dH_{i+1} + c_i dH_1, \qquad \text{with} \qquad H_{n+1} = 0\,. \tag{4.10}
$$

Here $N^* = P_0^{-1}P_1'$ and c_i are coefficients of the minimal characteristic polynomial of the recursion operator

$$\Delta_N(\lambda) = \Big(\det(N - \lambda\,\mathrm{I})\Big)^{1/2} = \lambda^n - (c_1\lambda^{n-1} + \cdots + c_n)\,, \tag{4.11}$$

which can be defined directly via the entries of the matrix $T(\lambda)$

$$\Delta_N(\lambda) = A(\lambda) + B(\lambda) - C(\lambda) - D(\lambda)\,. \tag{4.12}$$

Remark 2 Transformations (2.19) of the matrix $T(\lambda)$ give rise to canonical transformations in the phase space. As sequence tensor P_1' (4.9) coincides with tensor P_1 (4.3) after the following canonical transformation

$$p_1 \to p_1 + \mathrm{e}^{-q_1}, \qquad p_n \to p_n + \mathrm{e}^{q_n}\,,$$

which identifies coefficients c_i with integrals of motion for the open Toda lattice $c_i = -H_i^o$.

5 Integrable DST model

The integrable case of the DST (discrete self-trapping) model with n degrees of freedom was studied in [5]. It appears as a specialization of our basic model when several parameters vanish:

$$\beta = \gamma = \delta = 0 \qquad \text{and} \qquad Q_j = 0, \qquad j = 1, \ldots, n-1. \tag{5.1}$$

We also put $\alpha = 1$ and $\eta = -1$. In the lattice representation, the matrix $T(\lambda)$ (1.3) acquires the form

$$T(\lambda) = L_1(\lambda - c_1)\; L_2(\lambda - c_2)\; \cdots\; L_n(\lambda - c_n), \quad \text{with} \quad L_i(\lambda) = \begin{pmatrix} \lambda - q_i p_i & b q_i \\ -p_i & b \end{pmatrix}. \tag{5.2}$$

Here p_i, q_i are dynamical variables, whereas b and c_i are numbers entering into the Casimir function (1.4)

$$d(\lambda) = \det T(\lambda) = b^n(\lambda - c_1)(\lambda - c_2)\cdots(\lambda - c_n).$$

Substituting matrix $T(\lambda)$ (5.2) into the Sklyanin bracket (1.1) and into the brackets (4.7)-(4.8) one gets canonical brackets

$$\{p_i, q_j\}_0 = \delta_{ij}, \qquad \{q_i, q_j\}_0 = \{p_i, p_j\}_0 = 0, \qquad i, j = 1, \ldots, n\,.$$

and quadratic brackets

$$\begin{array}{llll} \{q_i, q_j\}_1 &= -Q_i Q_j, & \{q_i, p_j\}_1 &= Q_i P_j - c_i\,\delta_{ij}, \qquad i > j \\ \{p_i, p_j\}_1 &= -P_i P_j, & \{p_i, q_j\}_1 &= q_i p_j - b\,\delta_{i+1 j}\,, \end{array}$$

where $Q_1 = q_1 + 1$, $P_n = p_n + b$ and $Q_i = q_i$, $P_i = p_i$ for other values of index i.

As above the Hamiltonians H_i, $i = 1, \ldots, n$, from the $\mathrm{tr}T(\lambda) = \lambda^n + H_1\lambda^{n-1} + \ldots + H_n$ satisfy the Fröbenius relations (4.10). The two first Hamiltonians of the system are

$$\begin{aligned} H_1 &= -\sum_{i=1}^{n}(q_i p_i - c_i), \\ H_2 &= \sum_{i>j}(q_i p_i - c_i)(q_j p_j - c_j) - b\sum_{i=1}^{n} q_i p_{i+1}, \qquad p_{n+1} \equiv p_1. \end{aligned} \tag{5.3}$$

The Sklyanin variables (u_i, v_i), $i = 1, \ldots, n$, are introduced by the same formulae as for the Toda lattice, cf. (4.5) and (4.6).

6 The Goryachev-Chaplygin gyrostat

Let us consider the matrix $T(\lambda)$ introduced in [4]

$$T(\lambda) = \begin{pmatrix} \lambda^2 - 2\lambda J_3 - J_1^2 - J_2^2 & (x_1 + \mathrm{i}x_2)\lambda - x_3(J_1 + \mathrm{i}J_2) \\ (x_1 - \mathrm{i}x_2)\lambda - x_3(J_1 - \mathrm{i}J_2) & -x_3^2 \end{pmatrix}. \tag{6.1}$$

Substituting matrix (6.1) into the Sklyanin bracket (1.1) and brackets (2.3)-(2.4) at $\eta = 2\mathrm{i}$ one gets canonical Poisson tensor on the dual space of Euclidean algebra $e(3)$

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & x_3 & -x_2 \\ * & 0 & 0 & -x_3 & 0 & x_1 \\ * & * & 0 & x_2 & -x_1 & 0 \\ * & * & * & 0 & J_3 & -J_2 \\ * & * & * & * & 0 & J_1 \\ * & * & * & * & * & 0 \end{pmatrix} \tag{6.2}$$

and the following quadratic tensor

$$P_1 = \begin{pmatrix} 0 & -x_3^2 & x_3x_2 & -x_2J_1 & -x_2J_2 & x_3J_2 - 2x_2J_3 \\ * & 0 & -x_3x_1 & x_1J_1 & x_1J_2 & 2x_1J_3 - x_3J_1 \\ * & * & 0 & 0 & 0 & -x_1J_2 + x_2J_1 \\ * & * & * & 0 & -J_1^2 - J_2^2 & -J_3J_2 \\ * & * & * & * & 0 & J_1J_3 \\ * & * & * & * & * & 0 \end{pmatrix}. \tag{6.3}$$

These tensors satisfy equations (2.1) at any values of the Casimir functions

$$\mathcal{C}_1 = x_1^2 + x_2^2 + x_3^2, \qquad \mathcal{C}_2 = x_1J_1 + x_2J_2 + x_3J_3\,.$$

However, in the proposed method coefficients of $\det T = -\mathcal{C}_1\lambda^2 + x_3\,\mathcal{C}_2\,\lambda$ have to be the Casimir functions and, therefore, we have to put $\mathcal{C}_2 = 0$. As sequence, we have $\{A(\lambda), A(\mu)\}_1 = 0$ at $\mathcal{C}_2 = 0$ only.

Remark 3 Solving equations $P_0 dH_2^o = (P_1 + \alpha P_0)H_1^o$ at arbitrary values of $\mathcal{C}_{1,2}$ one gets

$$H_1^o = J_3, \qquad H_2^o = J_1^2 + J_2^2 + 2J_3^2 + \alpha J_3\,.$$

Here H_2^o is a kinetic part of the Hamiltonian for the Kowalevski gyrostat, which may be studied by using 2×2 Lax matrix $L(\lambda) = K_+ T(\lambda) K_- T^{-1}(-\lambda)$ [4]. The tensor P_1 (6.3) differs from the Poisson tensor for the Kowalevski gyrostat, which appears from the linear r-matrix algebra [14].

The 2×2 Lax matrix for the Goryachev-Chaplygin gyrostat looks like [8]

$$\widetilde{T}(\lambda) = \begin{pmatrix} \mathrm{e}^{\frac{q}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{q}{2}} \end{pmatrix} \begin{pmatrix} \lambda + 2J_3 + p & a \\ a & 0 \end{pmatrix} T(\lambda) \begin{pmatrix} \mathrm{e}^{-\frac{q}{2}} & 0 \\ 0 & \mathrm{e}^{\frac{q}{2}} \end{pmatrix}. \tag{6.4}$$

Here p, q are additional dynamical variables, a is an arbitrary number and $T(\lambda)$ is given by (6.1).

Substituting this matrix into the Sklyanin bracket (1.1) and into the brackets (4.7)-(4.8) one gets the compatible Poisson tensors on the extended phase space $e^*(3) \ltimes (p, q)$

$$\widetilde{P}_0 \equiv \begin{pmatrix} P_0 & W_0 \\ W_0^T & G_0 \end{pmatrix} = \left(\begin{array}{c|c} P_0 & \begin{matrix} 0 \\ \vdots \end{matrix} \\ \hline * & \begin{matrix} 0 & 2\mathrm{i} \\ * & 0 \end{matrix} \end{array}\right) \tag{6.5}$$

and

$$\widetilde{P}_1 \equiv \begin{pmatrix} P_1 & W_1 \\ W_1^T & G_1 \end{pmatrix} = \left(\begin{array}{c|c} P_1 & \begin{matrix} -2x_3J_2 + 2px_2 + 8x_2J_3 & -2x_2 \\ 2x_3J_1 - 2px_1 - 8x_1J_3 & 2x_1 \\ 2x_1J_2 - 2x_2J_1 & 0 \\ 2J_2(p + 3J_3) & -2J_2 + \mathrm{i}x_3\mathrm{e}^q \\ -2ax_3 - 2pJ_1 - 6J_1J_2 & 2J_1 - x_3\mathrm{e}^q \\ 2ax_2 & -\mathrm{i}(x_1 + \mathrm{i}x_2)\mathrm{e}^q \end{matrix} \\ \hline * & \begin{matrix} 0 & 2\mathrm{i}\big((x_1 - \mathrm{i}x_2)\mathrm{e}^q - 2J_3 - p - a\mathrm{e}^q\big) \\ * & 0 \end{matrix} \end{array}\right), \tag{6.6}$$

which satisfy equations (2.1) at $\mathcal{C}_2 = 0$ only. Here P_0 and P_1 are given by (6.2) and (6.3).

The Hamiltonians H_i from the $\mathrm{tr}\widetilde{T}(\lambda) = \lambda^3 + H_1\lambda + \lambda^2 H_2 + H_3$ are

$$\begin{aligned} H_1 &= p, \\ H_2 &= -\left(J_1^2 + J_2^2 + 4J_3^2 + 2pJ_3 - 2ax_1\right), \\ H_3 &= -(2J_3 + p)(J_1^2 + J_2^2) - 2ax_3J_1\,. \end{aligned}$$

The obtained tensors $\widetilde{P}_0$ and $\widetilde{P}_1$ are degenerate and, therefore, the Hamiltonians H_i reproduce the Fröbenius chain in the following form

$$\widetilde{P}_1 dH_i = \widetilde{P}_0 \left(dH_{i+1} + c_i dH_1\right), \qquad i = 1, 2, 3, \tag{6.7}$$

where $H_4 = 0$ and c_i are coefficients of the polynomial $\Delta_N(\lambda) = A(\lambda) + B(\lambda) - C(\lambda) - D(\lambda)$ (4.11)-(4.12).

At $p = \rho$ and $q = 0$ matrices G_0 (6.5) and G_1 (6.6) are (generically) non-degenerate. So, the Dirac procedure can reduce pencil $\widetilde{P}_0 + \lambda \widetilde{P}_1$ to a new Poisson pencil $\widetilde{P}_0^D + \lambda \widetilde{P}_1^D$ on $e^*(3)$ defined by

$$\widetilde{P}_k^D = P_k + \left(W_k\, G_k^{-1}\, W_k^T\right)_{p=\rho,\, q=0}\,, \qquad k = 0, 1.$$

Here $P_0 = \widetilde{P}_0^D$ is canonical Poisson tensor (6.2) and P_1 is given by (6.3). This reduction procedure preserves equations (6.7) for the reduced integrals of motion.

7 Conclusion

We present a family of compatible Poisson brackets (2.10), that includes the Sklyanin bracket, and prove that the Sklyanin variables are dual to the special Darboux-Nijenhuis coordinates associated with these brackets. The application of the r-matrix formalism is extremely useful here resulting in drastic reduction of the calculations for a whole set of integrable systems.

The construction can be generalized to other r-matrix algebras. Remind, if one substitutes $T(\lambda) = 1 + \varepsilon L(\lambda) + O(\varepsilon^2)$, $r = \varepsilon\, r$ into (1.1) and let $\varepsilon \to 0$ one gets a linear bracket. Then if $T(\lambda)$ satisfy the Sklyanin bracket (1.1), then the matrix $\mathcal{T}(\lambda) = T(\lambda) K_- T^{-1}(-\lambda)$ obeys to the reflection equation algebra [10]. The corresponding compatible brackets for the open generalized Toda lattices was considered in [15].

Moreover, the whole construction can immediately be transferred to the quantum case because r-matrices in (2.10) became dynamical matrices at $k > 1$ only.

We would like to thank I.V. Komarov and V.I. Inozemtsev for useful and interesting discussions. The research was partially supported by the RFBR grant 06-01-00140.

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