# Generalized N=4 supersymmetric Toda lattice hierarchy and N=4 superintegrable mapping

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#### Abstract

It is shown that the one-dimensional generalized N=4 supersymmetric Toda lattice (TL) hierarchy (nlin.Si/0311030) contains the N=4 super-KdV hierarchy with the first flow time in the role of space coordinate. Two different N=2 superfield forms of the generalized N=4 supersymmetric TL equation, which are useful when solving the N=4 super-KdV and (1,1)-GNLS hierarchies, are discussed.

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# 1 Introduction

The Toda lattice (TL) and its supersymmetric extensions being one of the most important families in the theory of integrable systems were the subject of many studies for the last decades. The remarkable property of the TL equations is that they are closely related to the famous differential integrable hierarchies of the NLS and KdV types. Recently, the generalized N=4 supersymmetric TL hierarchy, which contains the N=2 and N=4 supersymmetric TL equations as the result of appropriate reductions, was proposed [1]. Is is known that the N=2 supersymmetric TL equation serves as the symmetry transformation of the N=2 supersymmetric NLS hierarchy [2] while the N=4 supersymmetric TL equation is directly relevant to the N=4 supersymmetric KdV hierarchy [3, 4]. At the same time, the relevance of the generalized N=4 supersymmetric TL equation to the differential hierarchies has not been studied yet and the present paper addresses this problem.

One of the methods for finding differential hierarchies starting with the lattice equations is based on solving the appropriate symmetry equation [5, 6]. In this approach, a number of know and new N=2 supersymmetric hierarchies of differential equations were reproduced [2, 7, 8]. Another procedure by which one can extract differential hierarchy from the lattice one associated with the one-matrix model was proposed in [9] where it was demonstrated that the lattice hierarchy contains already the differential one with the first flow time  $t_1$  in the role of space coordinate. In such an approach, all the flows of the lattice hierarchy can simply be rewritten in the form of differential equations if one uses the first flow of the lattice hierarchy in order to express all the lattice fields via the lattice fields defined in the same lattice node. In the present paper, we apply this approach to the generalized N=4 supersymmetric TL hierarchy and demonstrate that the generalized N=4 supersymmetric TL equation forms the discrete symmetry of the N=4 supersymmetric KdV hierarchy.

The paper is organized as follows. In section 2, we recall the Lax-pair representation of the generalized N=4 supersymmetric TL hierarchy and consider two different reductions of its first flow which lead to the N=2 and N=4 supersymmetric TL equations. In section 3, we use the first flow of the the generalized N=4 supersymmetric TL hierarchy in order to rewrite all its flows in terms of fields defined in the same lattice node, which allows us to reproduce the component form of the N=4 supersymmetric KdV hierarchy in some new basis as well as its supersymmetric transformations and conservation laws. In section 4, we discuss two different N=2 superfield representations of the generalized N=4 supersymmetric TL equation which are useful when solving the (1,1)-GNLS and N=4 supersymmetric KdV hierarchy.

# 2 1D generalized fermionic Toda lattice hierarchy

In this section, we remind the Lax-pair formulation and the basic properties of the one-dimensional generalized N=4 supersymmetric TL hierarchy [1]. This hierarchy is generated by the following equation:

$$\partial_k L = [(L_+)^k, L], \tag{1}$$

for the infinite supermatrices

$$(L)_{i,j} = \delta_{i,j-2} + \gamma_i \delta_{i,j-1} + c_i \delta_{i,j} + \rho_i \delta_{i,j+1} + d_i \delta_{i,j+2}, \quad i \in \mathbb{Z}$$

$$\tag{2}$$

where  $\partial_k \equiv \partial/\partial t_k$ , the subscript + denotes the upper (including diagonal) triangular part of the matrix and the matrix entries  $c_i, d_i$  ( $\rho_i, \gamma_i$ ) are the bosonic (fermionic) lattice fields with the Grassmann parity 0 (1) and the length dimensions  $[d_i] = -2$ ,  $[c_i] = -1$ ,  $[\rho_i] = -3/2$ ,  $[\gamma_i] = -1/2$ ; *i* is the lattice index. Note, the supermatrix *L* is bosonic and its Grassmann parity is defined by the Grassmann parity of elements  $c_i$  on the main diagonal.

The first two flows originated from the Lax-pair representation (1) have the following explicit form:

$$\partial_1 d_i = d_i (c_i - c_{i-2}), \qquad \partial_1 c_i = d_{i+2} - d_i + \gamma_i \rho_{i+1} + \gamma_{i-1} \rho_i, 
\partial_1 \gamma_i = \rho_{i+2} - \rho_i, \qquad \partial_1 \rho_i = \rho_i (c_i - c_{i-1}) + d_{i+1} \gamma_i - d_i \gamma_{i-2}$$
(3)

and

$$\begin{aligned} \partial_{2}d_{i} &= d_{i}(d_{i+2} - d_{i-2} + c_{i}^{2} - c_{i-2}^{2} - \rho_{i-2}\gamma_{i-3} + \rho_{i-1}\gamma_{i-2} + \rho_{i}\gamma_{i-1} - \rho_{i+1}\gamma_{i}), \\ \partial_{2}c_{i} &= d_{i+2}(c_{i} + c_{i+2} + \gamma_{i}\gamma_{i+1}) - d_{i}(c_{i} + c_{i-2} + \gamma_{i-2}\gamma_{i-1}) \\ &\quad -\rho_{i}(\rho_{i-1} + \gamma_{i-1}(c_{i} + c_{i-1})) - \rho_{i+1}(\rho_{i+2} + \gamma_{i}(c_{i} + c_{i+1})), \\ \partial_{2}\gamma_{i} &= \rho_{i+2}(c_{i+1} + c_{i+2}) - \rho_{i}(c_{i} + c_{i-1}) + d_{i+3}\gamma_{i+2} + (d_{i+2} - d_{i+1})\gamma_{i} - d_{i}\gamma_{i-2}, \\ \partial_{2}\rho_{i} &= \rho_{i}(c_{i}^{2} - c_{i-1}^{2} + d_{i+2} - d_{i-1} - \rho_{i+1}\gamma_{i} - \rho_{i-1}\gamma_{i-2}) \\ &\quad -d_{i}(\rho_{i-2} + \gamma_{i-2}(c_{i-1} + c_{i-2})) + d_{i+1}(\rho_{i+2} + \gamma_{i}(c_{i} + c_{i+1})). \end{aligned}$$

Using the Lax pair representation (1), it is easy to derive the general expression for bosonic Hamiltonians, which are in involution, via the standard formula

$$H_{k} = \frac{1}{k} strL^{k} \equiv \frac{1}{k} \sum_{i=-\infty}^{\infty} (-1)^{i} (L^{k})_{ii}$$
(5)

with the Hamiltonian densities  $(-1)^i (L^k)_{ii}$  expected to satisfy the equation with respect to the evolution time  $t_s$ 

$$\partial_s((-1)^i(L^k)_{ii}) = \ell_{s,k,i} - \ell_{s,k,i-1} \equiv (\Delta \ell)_{s,k,i},\tag{6}$$

where  $\ell_{s,k,i}$  are polynomials of the lattice fields  $\{c_i, d_i, \gamma_i, \rho_i\}$ . In what follows we assume the zero boundary conditions at infinity for the lattice fields

$$\lim_{i \to \pm \infty} \{c_i, d_i, \gamma_i, \rho_i\} = 0 \tag{7}$$

in order the equation for the lattice conservation laws

$$\partial_s H_k = \sum_{i=-\infty}^{\infty} (\Delta \ell)_{s,k,i} = \lim_{i \to \infty} \ell_{s,k,i} - \lim_{i \to -\infty} \ell_{s,k,i} = 0$$
(8)

to be satisfied. Let us give here the explicit expressions for the first two lattice Hamiltonians obtained via formula (5)

$$H_1 = \sum_{i=-\infty}^{\infty} (-1)^i c_i, \qquad H_2 = \sum_{i=-\infty}^{\infty} (-1)^i (\frac{1}{2}c_i^2 + d_i + \rho_i \gamma_{i-1}), \qquad (9)$$

The remarkable feature of the first flow (3) is that it can be reduced to any of two 1D Toda lattice equations with extended supersymmetry known up to recently. First, the N=4 supersymmetric TL equation can be deduced by eliminating the field  $c_i$  from system (3) and transition to the new basis as follows:

$$d_i = g_i g_{i-1}, \quad \rho_i = g_i \gamma_i^-, \quad \gamma_i = \gamma_{i+1}^+. \tag{10}$$

In this basis eq.(3) takes the following form

$$\partial^{2} \ln g_{i} = g_{i+1}g_{i+2} - g_{i}(g_{i+1} - g_{i-1}) + g_{i-1}g_{i-2} + g_{i+1}\gamma_{i+1}^{+}\gamma_{i-1}^{-} - g_{i-1}\gamma_{i-1}^{+}\gamma_{i-1}^{-}, 
\partial\gamma^{\pm} = g_{i+1}\gamma_{i+1}^{\mp} - g_{i-1}\gamma_{i-1}^{\mp},$$
(11)

that is a component form of the N=4 supersimmetric TL equation.

Second, the N=2 supersymmetric TL equation comes from system (3) as a result of the following reduction. Let us introduce a new basis in the space  $\{d_i, c_i, \rho_i, \gamma_i\}$  which separates odd and even lattice nodes

$$a_{i} = c_{2i+1}, \quad b_{i} = d_{2i+1}, \quad \alpha_{i} = \gamma_{2i-1}, \quad \beta_{i} = \rho_{2i+1}, \\ \bar{a}_{i} = c_{2i}, \quad \bar{b}_{i} = d_{2i}, \quad \bar{\alpha}_{i} = -\gamma_{2i}, \quad \bar{\beta}_{i} = \rho_{2i}$$
(12)

and rewrite first flow (3) as follows:

$$\begin{aligned}
\partial_{1}b_{i} &= b_{i}(a_{i} - a_{i-1}), \quad \partial_{1}a_{i} = b_{i+1} - b_{i} + \beta_{i}\bar{\alpha}_{i} + \alpha_{i+1}\bar{\beta}_{i+1}, \\
\partial_{1}\bar{b}_{i} &= \bar{b}_{i}(\bar{a}_{i} - \bar{a}_{i-1}), \quad \partial_{1}\bar{a}_{i} = \bar{b}_{i+1} - \bar{b}_{i} + \beta_{i}\bar{\alpha}_{i} + \alpha_{i}\bar{\beta}_{i}, \\
\partial_{1}\alpha_{i} &= \beta_{i} - \beta_{i-1}, \quad \partial_{1}\beta_{i} = (a_{i} - \bar{a}_{i})\beta_{i} - b_{i}\alpha_{i} + \bar{b}_{i+1}\alpha_{i+1}, \\
\partial_{1}\bar{\alpha}_{i} &= \bar{\beta}_{i} - \bar{\beta}_{i+1}, \quad \partial_{1}\bar{\beta}_{i} = (\bar{a}_{i} - a_{i-1})\bar{\beta}_{i} - b_{i}\bar{\alpha}_{i} + \bar{b}_{i}\bar{\alpha}_{i-1}.
\end{aligned}$$
(13)

Now one can easily check that imposing the reduction constraints

$$\bar{b}_i = 0, \qquad \bar{a}_i = -\frac{\beta_i \bar{\beta}_i}{b_i} \tag{14}$$

on eq.(13) turns it into the following system of equations:

$$\partial b_{i} = b_{i}(a_{i} - a_{i-1}), \quad \partial a_{i} = b_{i+1} - b_{i} + \beta_{i}\bar{\alpha}_{i} + \alpha_{i+1}\beta_{i+1}, \\ \partial \beta_{i} = a_{i}\beta_{i} - b_{i}\alpha_{i}, \quad \partial\bar{\beta}_{i} = -a_{i-1}\bar{\beta}_{i} - b_{i}\bar{\alpha}_{i}, \\ \partial \alpha_{i} = \beta_{i} - \beta_{i-1}, \quad \partial\bar{\alpha}_{i} = \bar{\beta}_{i} - \bar{\beta}_{i+1}.$$
(15)

which is recognized as a component form of the the N=2 supersymetric TL equation.

We would like to emphasize that eqs.(11) and (15) are two different reductions of system (3) (or (13)) which are not related to each other, i.e. N=2 supersymmetric TL equation (15) cannot be obtained from N=4 supersymmetric TL equation (11) as a result of some reduction. Keeping in mind that both N=4 and N=2 TL equations can be deduced from eq.(3) we call eq.(3) the generalized N=4 supersymmetric TL equation.

The N=4 supersymmetry of eq.(3) in the basis (12) is realized by the following transformations:

$$\begin{aligned} \delta_{\epsilon} b_{i} &= \epsilon_{1} (u_{i} \bar{\beta}_{i} + \bar{u}_{i} \beta_{i}) - \epsilon_{2} (u_{i} \bar{\beta}_{i} - \bar{u}_{i} \beta_{i}) + \epsilon_{3} b_{i} (\alpha_{i} + \bar{\alpha}_{i}) + \epsilon_{4} b_{i} (\alpha_{i} - \bar{\alpha}_{i}), \\ \delta_{\epsilon} \bar{b}_{i} &= \epsilon_{1} (u_{i-1} \bar{\beta}_{i} + \bar{u}_{i} \beta_{i-1}) + \epsilon_{2} (u_{i-1} \bar{\beta}_{i} - \bar{u}_{i} \beta_{i-1}) + \epsilon_{3} \bar{b}_{i} (\alpha_{i} + \bar{\alpha}_{i-1}) + \epsilon_{4} \bar{b}_{i} (\alpha_{i} - \bar{\alpha}_{i-1}), \\ \delta_{\epsilon} a_{i} &= \epsilon_{1} (\bar{u}_{i+1} \alpha_{i+1} - u_{i} \bar{\alpha}_{i}) - \epsilon_{2} (\bar{u}_{i+1} \alpha_{i+1} - u_{i} \bar{\alpha}_{i}) + \epsilon_{3} (\beta_{i} - \bar{\beta}_{i+1}) + \epsilon_{4} (\beta_{i} + \bar{\beta}_{i+1}), \\ \delta_{\epsilon} \bar{a}_{i} &= \epsilon_{1} (\bar{u}_{i} \alpha_{i} - u_{i} \bar{\alpha}_{i}) - \epsilon_{2} (\bar{u}_{i} \alpha_{i} - u_{i} \bar{\alpha}_{i}) + \epsilon_{3} (\beta_{i} - \bar{\beta}_{i}) + \epsilon_{4} (\beta_{i} + \bar{\beta}_{i}), \\ \delta_{\epsilon} \beta_{i} &= \epsilon_{1} u_{i} (\bar{a}_{i} - a_{i}) + \epsilon_{2} u_{i} (\bar{a}_{i} - a_{i}) + \epsilon_{3} (b_{i} - \bar{b}_{i+1} - \beta_{i} \bar{\alpha}_{i}) + \epsilon_{4} (b_{i+1} - b_{i} + \beta_{i} \bar{\alpha}_{i}), \\ \delta_{\epsilon} \bar{\beta}_{i} &= \epsilon_{1} \bar{u}_{i} (a_{i-1} - \bar{a}_{i}) - \epsilon_{2} \bar{u}_{i} (a_{i-1} - \bar{a}_{i}) + \epsilon_{3} (b_{i} - \bar{b}_{i} - \bar{\beta}_{i} \alpha_{i}) + \epsilon_{4} (b_{i} - \bar{b}_{i} - \bar{\beta}_{i} \alpha_{i}), \\ \delta_{\epsilon} \alpha_{i} &= \epsilon_{1} (u_{i-1} - u_{i}) + \epsilon_{2} (u_{i-1} - u_{i}) + \epsilon_{3} (a_{i-1} - \bar{a}_{i}) + \epsilon_{4} (\bar{a}_{i} - a_{i-1}), \\ \delta_{\epsilon} \bar{\alpha}_{i} &= \epsilon_{1} (\bar{u}_{i+1} - \bar{u}_{i}) + \epsilon_{2} (\bar{u}_{i} - \bar{u}_{i+1}) + \epsilon_{3} (\bar{a}_{i} - a_{i}) + \epsilon_{4} (\bar{a}_{i} - a_{i}), \end{aligned}$$

where  $\epsilon_k$  (k = 1, 2, 3, 4) are the corresponding fermionic infinitesimal parameters. Note that transformations corresponding to the parameters  $\epsilon_1$  and  $\epsilon_2$  are nonlocal with respect to the lattice indices and they are expressed via composite fields  $u_i, \bar{u}_i$ 

$$u_i \equiv \prod_{k=0}^{\infty} \frac{b_{i-k}}{\overline{b}_{i-k}}, \qquad \bar{u}_i \equiv \prod_{k=0}^{\infty} \frac{\overline{b}_{i-k}}{\overline{b}_{i-k-1}}$$
(17)

which obey the following equations and N=4 supersymmetric transformations

$$\partial_1 u_i = u_i (a_i - \bar{a}_i), \quad \partial_1 \bar{u}_i = \bar{u}_i (\bar{a}_i - a_{i-1}),$$
  

$$\delta_\epsilon u_i = \epsilon_1 \beta_i - \epsilon_2 \beta_i + \epsilon_3 u_i \bar{\alpha}_i - \epsilon_4 u_i \bar{\alpha}_i,$$
  

$$\delta_\epsilon \bar{u}_i = \epsilon_1 \bar{\beta}_i + \epsilon_2 \bar{\beta}_i + \epsilon_3 \bar{u}_i \alpha_i + \epsilon_4 \bar{u}_i \alpha_i.$$
(18)

It can be easily checked that the above transformations indeed realize N=4 supersymmetry, i.e., that their commutators for any field  $q_i \equiv \{a_i, \bar{a}_i, b_i, \bar{b}_i, \beta_i, \beta_i, \alpha_i, \bar{\alpha}_i\}$  give

$$[\delta_{\epsilon}, \delta_{\tilde{\epsilon}}]q_i = 2\sum_{k=1}^4 (-1)^k \tilde{\epsilon}_k \epsilon_k \partial_1 q_i.$$
<sup>(19)</sup>

### 3 From the Toda lattice hierarchy to the differential one

In this section, we show that first flow (13) of the generalized N=4 supersymmetric TL hierarchy underlies the N=4 supersymmetric hierarchy of differtial equations which is related to N=4 supersymmetric KdV hierarchy.

Our goal is to construct the hierarchy of differential equations for which system (13) works as the discrete symmetry mapping connecting its different solutions. With this aim one can deduce the appropriate symmetry equation [5, 6] and try to solve it. However, in the case at hand we deal with all the flows of the lattice hierarchy (1), which allows us to apply another approach [9] and simply rewrite all the lattice flows via the lattice fields defined in the same lattice node. Indeed, one can use first flow (13) in order to express all the lattice fields via the lattice fields defined in the *i*-th lattice node as follows:

$$\begin{array}{ll}
\begin{aligned}
a_{i-1} &= a_i - (\log b_i)', & \beta_{i+1} &= \beta_i + \alpha'_{i+1}, \\
\bar{a}_{i-1} &= \bar{a}_i - (\log \bar{b}_i)', & \bar{\beta}_{i-1} &= \bar{\beta}_i + \bar{\alpha}'_{i-1}, \\
\beta_{i-1} &= \beta_i - \alpha'_i, & b_{i-1} &= b_i + \alpha_i \bar{\beta}_i - \bar{\alpha}_{i-1} \beta_{i-1} - a'_{i-1}, \\
\bar{\beta}_{i+1} &= \bar{\beta}_i - \bar{\alpha}'_i, & b_{i-1} &= b_i + \alpha_i \bar{\beta}_i - \bar{\alpha}_{i-1} \beta_{i-1} - a'_{i-1}, \\
\bar{b}_{i+1} &= \bar{a}'_i + \bar{b}_i + \bar{\alpha}_i \beta_i - \alpha_i \bar{\beta}_i, & \bar{a}_{i+1} &= \bar{a}_i + (\log b_{i+1})', \\
\alpha_{i+1} &= (\beta'_i + b_i \alpha_i - \beta_i (a_i - \bar{a}_i))/\bar{b}_{i+1}, & \bar{\alpha}_{i+1} &= (\bar{b}_{i+1} \bar{\alpha}_i + \bar{\beta}_{i+1} (\bar{a}_{i+1} - a_i) - \bar{\beta}'_{i+1})/b_{i+1}, \\
\bar{\alpha}_{i-1} &= (\bar{\beta}'_i + b_i \bar{\alpha}_i + \bar{\beta}_i (a_{i-1} - \bar{a}_i))/\bar{b}_i, & \alpha_{i-1} &= (\bar{b}_i \alpha_i + \beta_{i-1} (a_{i-1} - \bar{a}_{i-1}) - \beta'_{i-1})/b_{i-1}, \\
b_{i+1} &= b_i - \alpha_{i+1} \bar{\beta}_{i+1} + \bar{\alpha}_i \beta_i + a'_i, & \bar{b}_{i-1} &= -\bar{a}'_{i-1} + \bar{b}_i - \bar{\alpha}_{i-1} \beta_{i-1} + \alpha_{i-1} \bar{\beta}_{i-1},
\end{aligned}$$
(20)

where we denote the  $t_1$ -derivative by the sign '. With the help of eq.(20) all the flows of the lattice hierarchy (1) being rewritten in basis (12) can be expressed in the terms of the lattice fields and their derivatives defined in the same lattice node. Thus, one can verify that second flow (4) turns into the following system of differential equations:

$$\begin{aligned}
\partial_{2}a &= (a' + a^{2} + 2(b - \beta\bar{\alpha}))', \\
\partial_{2}\bar{a} &= (\bar{a}' + \bar{a}^{2} + 2(\bar{b} + \bar{\beta}\alpha))', \\
\partial_{2}b &= -b'' + 2(ab)' + 2b(\bar{\beta}\alpha - \beta\bar{\alpha}), \\
\partial_{2}\bar{b} &= -\bar{b}'' + 2(\bar{a}\bar{b})' + 2(\bar{b}\bar{\beta}\alpha + (\alpha' - \beta)(\bar{\beta}' + b\bar{\alpha} + \bar{\beta}(a - \bar{a} - (\log b)'))), \\
\partial_{2}\beta &= \beta'' + 2(\bar{a}\beta + b\alpha)', \\
\partial_{2}\bar{\beta} &= -\bar{\beta}'' + 2(\bar{a}\beta - b\bar{\alpha})', \\
\partial_{2}\alpha &= -\alpha'' + 2(\beta' + \bar{a}\beta + \alpha(b - \bar{b}) + (\alpha' - \beta)(a - (\log b)')), \\
\partial_{2}\bar{\alpha} &= \bar{\alpha}'' - 2(\bar{\beta}' - a\bar{\alpha}' + \bar{\alpha}(b - \bar{b} - \bar{a}') + \bar{\beta}(a - \bar{a} - \alpha\bar{\alpha})),
\end{aligned}$$
(21)

where we omit the lattice index *i*. Therefore, one can conclude that using substitutions (20) one can rewrite all the flows of N=4 TL hierarchy (1) in the form of two-dimensional differential equations, i.e., pass from the lattice hierarchy to the differential one. The composite lattice fields (17) which are nonlocal with respect to lattice indices in this case become nonlocal with respect to time coordinate  $t_1$  and take the following form:

$$u_i = exp(\partial^{-1}(a_i - \bar{a}_i)) := e^{\Delta}, \quad \bar{u}_i = b_i exp(-\partial^{-1}(a_i - \bar{a}_i)) := b_i e^{-\Delta}$$
 (22)

The same procedure being applied to eq.(16) allows one to find the explicit expressions for the N=4 supersimmetric transformations of the new differential hierarchy which read

$$\delta_{\epsilon}a = \epsilon_1((b\alpha - a\beta + \bar{a}\beta + \beta')e^{-\Delta} - \bar{\alpha}e^{\Delta}) + \epsilon_2((a\beta - b\alpha - \bar{a}\beta - \beta')e^{-\Delta} - \bar{\alpha}e^{\Delta})$$

$$+\epsilon_{3}(\beta - \bar{\beta} + \bar{\alpha}') + \epsilon_{4}(\beta + \bar{\beta} - \bar{\alpha}'),$$

$$\delta_{\epsilon}\bar{a} = \epsilon_{1}(b\,\alpha e^{-\Delta} - \bar{\alpha}e^{\Delta}) + \epsilon_{2}(-b\,\alpha e^{-\Delta} - \bar{\alpha}e^{\Delta}) + \epsilon_{3}(\beta - \bar{\beta}) + \epsilon_{4}(\beta + \bar{\beta}),$$

$$\delta_{\epsilon}b = \epsilon_{1}(\bar{\beta}\,e^{\Delta} + b\,\beta e^{-\Delta}) + \epsilon_{2}(\bar{\beta}\,e^{\Delta} - b\,\beta e^{-\Delta}) + \epsilon_{3}b(\alpha + \bar{\alpha}) + \epsilon_{4}b(\alpha - \bar{\alpha}),$$

$$\delta_{\epsilon}\bar{b} = \epsilon_{1}(\bar{b}\,\bar{\beta}\,e^{\Delta}/b + b\,(\beta - \alpha')e^{-\Delta}) + \epsilon_{2}(\bar{b}\,\bar{\beta}\,e^{\Delta}/b - b\,(\beta - \alpha')e^{-\Delta}) + \epsilon_{3}(\bar{b}\,\alpha + b\,\bar{\alpha} + \bar{\beta}(a - \bar{a} - (\log b)') + \bar{\beta}') + \epsilon_{4}(\bar{b}\,\alpha - b\,\bar{\alpha} + \bar{\beta}(\bar{a} - a + (\log b)') - \bar{\beta}'),$$

$$\delta_{\epsilon}\alpha = (\epsilon_{1} + \epsilon_{2})(\bar{b}/b - 1)e^{\Delta} + (\epsilon_{3} - \epsilon_{4})(a - \bar{a} - (\log b)'),$$

$$\delta_{\epsilon}\beta = (\epsilon_{1} - \epsilon_{2})(\bar{b} - b - \beta\bar{\alpha} + \bar{\beta}\alpha + \bar{a}')e^{-\Delta} + (\epsilon_{3} + \epsilon_{4})(\bar{a} - a),$$

$$\delta_{\epsilon}\beta = (\epsilon_{1} - \epsilon_{2})(b(a - \bar{a}) - b')e^{-\Delta} + (\epsilon_{3} + \epsilon_{4})(b - \bar{b} - \bar{\beta}\alpha).$$
(23)

Note that the nonlocality of the supersymmetric transformations corresponding to the parameters  $\epsilon_1$  and  $\epsilon_2$  originates from the nonlocality of the composite lattice fields (17).

The main property of integrable hierarchy is that it possesses an infinite number of conservation laws. From eq.(21) one can find the first two conservation laws which have the following simple form:

$$\mathcal{H}_1 = \int dxa, \qquad \bar{\mathcal{H}}_1 = \int dx\bar{a}.$$
 (24)

Now let us show how all bosonic conservation laws of the new N=4 supersymmetric differential hierarchy can be produced from the lattice Hamiltonians (5). By construction the densities  $(-1)^i (L^k)_{ii}$  are conserved which means

$$\partial_1 \int dx (-1)^i (L^k)_{ii} = \int dx (\ell_{1,k,i} - \ell_{1,k,i-1}) = 0, \qquad (25)$$

where we rename  $t_1$  by x and take into consideration the boundary conditions (7) and eq.(6). From (25) one can obtain the relation which connects the integrals defined in the neighboring lattice nodes and using which one can write

$$\int dx \ell_{1,k,i} = \int dx \ell_{1,k,i-1} = \int dx \lim_{i \to -\infty} \ell_{1,k,i} = 0,$$
(26)

where the boundary conditions (7) are taken into consideration again. Thus, one can conclude that  $\ell_{1,k,i}$  is the conserved density and

$$\mathcal{H}_{k+1} = \int dx \ell_{1,k,i} \tag{27}$$

are the conservation laws,  $\partial_s \mathcal{H}_k = 0$ . In such a way, using eq.(20) in order to express all the fields entering into the density  $\ell_{1,k,i}$  in the same lattice node we obtain from Hamiltonians (9) the next two conservation laws of the differential N=4 supersymmetric hierarchy which read

$$\mathcal{H}_2 = \int dx (b - \bar{b} - \bar{\beta}\alpha), \qquad \mathcal{H}_3 = \int dx (ba - \bar{b}\bar{a} - b\alpha\bar{\alpha} - \bar{a}\bar{\beta}\alpha + \beta\bar{\beta}).$$
(28)

We have no independent formulation of the new differential hierarchy so far. All its flows, the supersymmetry transformations as well as the conservation laws can be generated only with the help of the substitutions (20) starting with the corresponding lattice counterparts. In order to give an independent formulation of the new hierarchy, one can try to find its Hamiltonian structure or construct its Lax-pair representation. But first of all, to understand what kind of hierarchy we deal with, let us consider the bosonic limit of its second flow (21). It is the system of two decoupled NLS equations

$$\partial_2 a = (a' + a^2 + 2b)', 
\partial_2 b = (-b' + 2ab)'$$
(29)

and the same equations for the fields  $\bar{a}$  and  $\bar{b}$ . The set of equations (29) forms the bosonic limit of the second flow of the a=4, N=2 supersymmetric KdV hierarchy. Keeping in mind that the new hierarchy possesses N=4 supersymmetry one can expect that it is closely related to the N=4 KdV hierarchy. It turns out that such a relation indeed exists. After passing to the new basis  $\{u, v, r, s, \xi, \bar{\xi}, \eta, \bar{\eta}\}$  in the space of fields  $\{b, \bar{b}, a, \bar{a}, \alpha, \bar{\alpha}, \beta, \bar{\beta}\}$  defined by the following transformations:

$$u = b - \bar{b} - a' + (\log b)'' + \alpha \bar{\beta}, \quad v = a - (\log b)', \quad r = b \ e^{-\Delta},$$
  

$$s = -\bar{b} \ e^{\Delta}/b, \quad \bar{\xi} = -\bar{\beta}, \quad \xi = \beta - \alpha', \quad \eta = \alpha b \ e^{-\Delta},$$
  

$$\bar{\eta} = ((\bar{a} - a + (\log b)')\bar{\beta} - \bar{\beta}' - b\bar{\alpha}) \ e^{\Delta}/b$$
(30)

one can rewrite the second flow (21) as follows:

$$\partial_{2}u = (-u' + 2uv + 2r's - 2\xi\bar{\xi} + 2\eta\bar{\eta})', \quad \partial_{2}\xi = (-\xi' - 2s\eta + 2v\xi)', \\ \partial_{2}v = (v' + v^{2} + 2u - 2rs)', \quad \partial_{2}\bar{\xi} = (\bar{\xi}' - 2r\bar{\eta} + 2v\bar{\xi})', \\ \partial_{2}r = r'' - 2ur + 2vr' + 2\eta\bar{\xi}, \quad \partial_{2}\eta = (\eta' + 2r\xi + 2v\eta)', \\ \partial_{2}s = -s'' + 2us + 2(vs)' - 2\xi\bar{\eta}, \quad \partial_{2}\bar{\eta} = (-\bar{\eta}' + 2s\bar{\xi} + 2v\bar{\eta})'.$$
(31)

The set of equations (31) represents the component form of the second flow of the a=4, N=4 supersymmetric KdV hierarchy [10, 11]. In [4], it was demonstrated that the a=4, N=4 super-KdV hierarchy as well as the a=-2, N=4 super-KdV one can be reproduced as a result of different reductions of the N=4 Toda-KdV hierarchy written in terms of two constrained N=4 superfields. The component form of the second flow of the a=-2, N=4 KdV hierarchy is <sup>1</sup>

$$\begin{aligned}
\partial_{2}\tilde{u} &= (\tilde{r}'' + \tilde{u}(\tilde{r} + 3\tilde{s}) - \tilde{v}\tilde{r}' - 3\tilde{\xi}\tilde{\eta} + \bar{\xi}\tilde{\eta})', \\
\partial_{2}\tilde{v} &= \tilde{s}'' - \tilde{r}'' + 2\tilde{u}(\tilde{r} - \tilde{s}) + \tilde{v}(\tilde{r} + \tilde{s})' + \tilde{v}'(\tilde{s} + 3\tilde{r}) + 2(\tilde{\xi}\tilde{\eta} + \tilde{\xi}\tilde{\eta}), \\
\partial_{2}\tilde{s} &= \tilde{u}' + \tilde{v}'' - \tilde{u}\tilde{v} - \tilde{v}\tilde{v}' + \tilde{s}'(\tilde{r} + 3\tilde{s}) - \tilde{\eta}\tilde{\eta} + \tilde{\xi}\tilde{\xi}, \\
\partial_{2}\tilde{r} &= \tilde{u}' + \tilde{u}\tilde{v} + \tilde{r}'(\tilde{s} + 3\tilde{r}) + \tilde{\eta}\tilde{\eta} - \tilde{\xi}\tilde{\xi}, \\
\partial_{2}\tilde{\xi} &= (-\tilde{\eta}' + \tilde{v}\tilde{\eta} + \tilde{\xi}(\tilde{r} + 3\tilde{s}))', \quad \partial_{2}\tilde{\xi} = (\tilde{\eta}' + \tilde{v}\tilde{\eta} + \tilde{\xi}(\tilde{s} + 3\tilde{r}))', \\
\partial_{2}\tilde{\eta} &= (-\tilde{\xi}' - \tilde{v}\tilde{\xi} + \tilde{\eta}(\tilde{s} + 3\tilde{r}))', \quad \partial_{2}\tilde{\eta} = (\tilde{\xi}' - \tilde{v}\tilde{\xi} + \tilde{\eta}(\tilde{r} + 3\tilde{s}))', \end{aligned}$$
(32)

<sup>&</sup>lt;sup>1</sup> Equation (32) corresponds to the second flow of the a=-2, b=-6, N=4 super-KdV hierarchy, according to the notation of [11]; the system (31) corresponds to the second flow of the a=4, b=0, N=4 super-KdV hierarchy.

One can easily verify that the component forms of the N=4 supersymmetric a=-2 and a=4 KdV hierarchies are related to each other by the following simple relation:

$$\tilde{u} = u + s' + v'/2 + r'/4, \quad \tilde{v} = -2s - r/2, \quad \tilde{r} = v/2 - s + r/4, \quad \tilde{s} = v/2 + s - r/4, 
\tilde{\xi} = \xi - \eta/2, \quad \tilde{\bar{\xi}} = \bar{\xi}/2 - \bar{\eta}, \quad \tilde{\eta} = \eta/2 + \xi, \quad \tilde{\bar{\eta}} = \bar{\eta} + \bar{\xi}/2.$$
(33)

Thus, one can conclude that the differential hierarchy deduced from the generalized N=4 supersymmetric TL hierarchy (1) with the help of substitutions (20) is the component form of the N=4 super-KdV hierarchy. To finish this section, we note that transformations (30) are invertible

$$a = v + (\log(u + v' - rs - \bar{\xi}\eta/r))', \quad b = u + v' - rs - \bar{\xi}\eta/r, \bar{a} = v + (\log r)', \quad \bar{b} = -rs, \quad \beta = \xi + (\eta/r)', \quad \bar{\beta} = -\bar{\xi}, \alpha = \eta/r, \quad \bar{\alpha} = \frac{\bar{\xi}' - \bar{\xi}(\log r)' - r\bar{\eta}}{u + v' - rs - \bar{\xi}\eta/r},$$
(34)

and relations (34) being substituted into eq.(13) give the discrete symmetry mapping connecting different solutions of the a=4, N=4 super-KdV hierarchy.

#### 4 From the differential hierarchy to the lattice one

In the previous section, we demonstrated how differential N=4 supersymmetric KdV hierarchy can be recovered from the generalized N=4 supersymmetric TL hierarchy (1). In this section, we resolve the inverse problem, i.e., show how TL hierarchy (1) can be reproduced from the differential one.

First of all, let us introduce the (1,1)-Generalized Nonlinear Schrödinger ((1,1)-GNLS) hierarchy [12]. All its flows can be deduced from the Lax-pair representation

$$\partial_k L = [(L^k)_{\ge 1}, L] \tag{35}$$

for the Lax operator

$$L = \partial - 1/2(F_a \overline{F}_a + F_a \overline{D} \partial^{-1} [D\overline{F}_a]), \qquad (36)$$

where  $F_a(X)$  and  $\overline{F}_a(X)$  are chiral and antichiral N=2 superfields

$$DF_a(X) = 0, \quad \overline{D} \ \overline{F}_a(X) = 0,$$
(37)

respectively, which are bosonic for a = 1 and fermionic for a = 2;  $X = (x, \theta, \overline{\theta})$  is a coordinate of N=2 superspace and  $D, \overline{D}$  are the N=2 supersymmetric fermionic covariant derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2}\bar{\theta}\frac{\partial}{\partial x}, \quad \overline{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial x}, \quad \{D, \overline{D}\} = -\frac{\partial}{\partial x}, \quad D^2 = \overline{D}^2 = 0$$
(38)

In (35) the subscript  $\geq 1$  means the sum of purely derivative terms of the operator  $L^k(k > 0)$ and for k = 2 eq.(35) gives

$$\partial_2 F_a = F_a'' + D(F_b \overline{F}_b \overline{D} F_a), 
\partial_2 \overline{F}_a = -\overline{F}_a'' + \overline{D}(F_b \overline{F}_b D \overline{F}_a),$$
(39)

where summation over repeated indices is understood. The set of equations (39) forms the (1,1)-GNLS equation which is related to the a=4, N=4 supersymmetric KdV hierarchy. In [13], it was demonstrated that the passing to the new basis

$$\mathcal{J} = -\frac{1}{2} \sum_{k=1}^{2} F_k \overline{F}_k - \frac{D\overline{F}_2'}{D\overline{F}_2}, \qquad \mathcal{F} = -\frac{1}{2} F_1 D\overline{F}_2, \qquad \overline{\mathcal{F}} = -\overline{D} D\left(\frac{\overline{F}_1}{D\overline{F}_2}\right) \tag{40}$$

establishes the relationship between these hierarchies. It is easy to show that the new superfields  $\mathcal{J}, \mathcal{F}, \overline{\mathcal{F}}$  satisfy, as a consequence of eq.(39), the following set of equations:

$$\partial_2 \mathcal{J} = ([D, \overline{D}] \mathcal{J} + \mathcal{J}^2 - 2\mathcal{F}\overline{\mathcal{F}})', \partial_2 \mathcal{F} = \mathcal{F}'' - 2D\overline{D}(\mathcal{J}\mathcal{F}), \qquad \partial_2 \overline{\mathcal{F}} = -\overline{\mathcal{F}}'' - 2\overline{D}D(\mathcal{J}\overline{\mathcal{F}}),$$
(41)

which is recognized as the second flow of the a=4, N=4 supersymmetric KdV hierarchy [11]. Let us note that the N=2 superfield form of the second flow (41) is related to its component form (31) as follows:

$$u = 1/2([D,\overline{D}]\mathcal{J} - \mathcal{J}')\Big|, \quad v = \mathcal{J}\Big|, \quad \xi = \overline{D}\mathcal{J}\Big|, \quad \bar{\xi} = D\mathcal{J}\Big|,$$
  
$$r = \mathcal{F}\Big|, \quad s = \overline{\mathcal{F}}\Big|, \quad \bar{\eta} = D\overline{\mathcal{F}}\Big|, \quad \eta = \overline{D}\mathcal{F}\Big|, \quad (42)$$

where | means the  $(\theta, \bar{\theta}) \rightarrow 0$  limit. For completeness, we give here the N=2 superfield form of the second flow of the a=-2, N=4 super-KdV equation (32)

$$\partial_{2}\tilde{\mathcal{J}} = (\tilde{\overline{\mathcal{F}}}' - \tilde{\mathcal{F}}' + \tilde{\mathcal{J}}(\tilde{\mathcal{F}} + \bar{\overline{\mathcal{F}}}))' - 2\overline{D}D(\tilde{\mathcal{J}}\tilde{\mathcal{F}}) - 2D\overline{D}(\tilde{\mathcal{J}}\bar{\overline{\mathcal{F}}}),$$
  

$$\partial_{2}\tilde{\mathcal{F}} = D\overline{D}(\tilde{\mathcal{J}}' + 1/2\tilde{\mathcal{J}}^{2} - \tilde{\mathcal{F}}\bar{\overline{\mathcal{F}}} - 3/2\tilde{\mathcal{F}}^{2}),$$
  

$$\partial_{2}\bar{\overline{\mathcal{F}}} = \overline{D}D(-\tilde{\mathcal{J}}' + 1/2\tilde{\mathcal{J}}^{2} - \tilde{\mathcal{F}}\bar{\overline{\mathcal{F}}} - 3/2\bar{\overline{\mathcal{F}}}^{2}),$$
(43)

where the component content of the superfields is defined in the same way as in eq.(42).

The precise analysis shows that in addition to relation (40) these exists at least one more relation

$$\mathcal{J} = -\frac{1}{2} \sum_{k=1}^{2} F_k \overline{F}_k + \frac{\overline{D} F_2'}{\overline{D} F_2} + \left(\frac{F_2 \overline{D} F_1}{F_1 \overline{D} F_2}\right)', \qquad \overline{\mathcal{F}} = -\overline{D} \left(\frac{F_2}{F_1}\right),$$
$$\mathcal{F} = D \left(\frac{1}{\overline{D} F_2} \left(\overline{D} \left(-F_1' + \frac{1}{4} F_1^2 \overline{F}_1 + \frac{F_1' F_2 \overline{D} F_1}{F_1 \overline{D} F_2}\right) + \frac{1}{2} F_2 \overline{D} F_1\right)\right)$$
(44)

which connects the (1, 1)-GNLS and a=4, N=4 super-KdV hierarchies. Supplying the supefields  $F_a, \overline{F}_a$  in eqs. (40) and (44) with the lattice index *i* and *i* - 1, respectively, and equating the corresponding superfields  $\mathcal{J}, \mathcal{F}, \overline{\mathcal{F}}$  beloning to relations (40) and (44), we obtain the mapping

$$\frac{1}{2} \sum_{k=1}^{2} (F_{k,i-1}\overline{F}_{k,i-1} - F_{k,i}\overline{F}_{k,i}) = \left(\log(\overline{D}F_{2,i-1}D\overline{F}_{2,i}) + \frac{F_{2,i-1}\overline{D}F_{1,i-1}}{F_{1,i-1}\overline{D}F_{2,i-1}}\right)', \\
\overline{D}D\left(\frac{\overline{F}_{1,i}}{D\overline{F}_{2,i}}\right) = \overline{D}\left(\frac{F_{2,i-1}}{F_{1,i-1}}\right), \\
D\left(\frac{1}{\overline{D}F_{2,i-1}}\left(\overline{D}\left(F_{1,i-1}' - \frac{1}{4}F_{1,i-1}^{2}\overline{F}_{1,i-1} - \frac{F_{1,i-1}'F_{2,i-1}\overline{D}F_{1,i-1}}{F_{1,-1}\overline{D}F_{2,i-1}}\right) - \frac{1}{2}F_{2,i-1}\overline{D}F_{1,i-1}\right)\right) = \frac{1}{2}F_{1,i}D\overline{F}_{2,i}, \quad (45)$$

that acts like the discrete symmetry transformation of the (1,1)-GNLS hierarchy and has significant importance in our consideration. Namely, one can demonstrate that all the flows of the generalized N=4 supersymmetric TL hierarchy (1) can be recovered from the corresponding flows of the (1,1)-GNLS hierarchy with the help of the mapping (45).

Let us introduce the components of the N=2 superfields entering into eq.(45) as follows:

$$g_{i} = F_{1,i} \Big|, \quad \bar{g}_{i} = \overline{F}_{1,i} \Big|, \quad f_{i} = \overline{D}F_{2,i} \Big|, \quad \bar{f}_{i} = D\overline{F}_{2,i} \Big|,$$
$$\chi_{i} = F_{2,i} \Big|, \quad \bar{\chi}_{i} = \overline{F}_{2,i} \Big|, \quad \zeta_{i} = \overline{D}F_{1,i} \Big|, \quad \bar{\zeta}_{i} = D\overline{F}_{1,i} \Big|, \quad (46)$$

where the fields  $g_i, \bar{g}_i, f_i, \bar{f}_i (\chi_i, \bar{\chi}_i, \zeta, \bar{\zeta}_i)$  are bosonic (fermionic) ones, and define the following relations connecting these component fields with the fields of the generalized N=4 supersymmetric TL hierarchy in basis (12):

$$a_{i} = -\frac{1}{2} (g_{i}\bar{g}_{i} + \chi_{i}\bar{\chi}_{i}) + \left(\log f_{i} - \frac{\zeta_{i}\chi_{i}}{g_{i}f_{i}}\right)', \quad b_{i} = \frac{1}{2} (\frac{\bar{f}_{i}\zeta_{i}\chi_{i}}{g_{i}} - f_{i}\bar{f}_{i}),$$

$$\bar{a}_{i} = -\frac{1}{2} (g_{i}\bar{g}_{i} + \chi_{i}\bar{\chi}_{i}) + (\log g_{i})', \quad \bar{b}_{i} = \frac{1}{2} (\frac{g_{i}\bar{\zeta}_{i}\bar{\chi}_{i}'}{\bar{f}_{i}} + g_{i}\bar{g}_{i}'),$$

$$\beta_{i} = \frac{1}{2} (f_{i}\bar{\chi}_{i} + \bar{g}_{i}\zeta_{i}) - \left(\frac{\zeta_{i}}{g_{i}}\right)', \quad \bar{\beta}_{i} = \frac{1}{2} (\bar{f}_{i}\chi_{i} - g_{i}\bar{\zeta}_{i}),$$

$$\alpha_{i} = \frac{\bar{\chi}_{i}'}{\bar{f}_{i}} - \frac{\zeta_{i}}{g_{i}}, \quad \bar{\alpha}_{i} = \frac{1}{f_{i}} (\chi_{i}' - \chi_{i}(\log g_{i})' + \frac{\zeta_{i}\chi_{i}\chi_{i}'}{g_{i}f_{i}}).$$
(47)

Now it is a matter of straightforward calculations to verify that the component form of the mapping (45) being rewritten in basis (47) is equivalent to the generalized N=4 supersymmetric TL equation (13). Therefore, eq.(45) represents the N=2 superfild form of this equation. Moreover, one can verify that fields  $\{a_i, \bar{a}_i, b_i, \bar{b}_i, \chi_i, \bar{\chi}_i, \zeta_i, \bar{\zeta}_i\}$  defined by eq.(47) satisfy, as a consequence of eqs.(39) and (46), the equations of the second flow (21) of the N=4 super-KdV hierarchy. Eliminating further the x-derivatives from eq.(21) with the help of eq.(13) one

obtains the second flow (4) of the generalized N=4 supersymmetric TL hierarchy in the basis (12). It is clear that the same procedure allows one to reproduce any flow of the generalized N=4 supersymmetric TL hierarchy (1) from the corresponding flow of the (1,1)-GNLS hierarchy. Thus, we can reproduce the generalized N=4 supersymmetric TL hierarchy starting with (1,1)-GNLS hierarchy and mapping (45).

Note that in a similar approach the N=2 supersymmetric TL hierarchy was constructed in [14]. The N=2 supersymmetric TL hierarchy can be recovered from the generalized N=4 supersymmetric TL hierarchy as a result of the reduction with the reduction constraints (14). The set of equations (15) is exactly the first flow of this N=2 supersymmetric TL hierarchy. The N=4 supersymmetric differential hierarchy in the basis (12) can also be reduced to the N=2 supersymmetric one by imposing the following reduction constraints:

$$\bar{b} = 0, \quad \bar{a} = -\frac{\beta\beta}{b}, \quad \alpha = \frac{1}{b}(a\beta - \beta'), \quad \bar{\alpha} = \frac{1}{b}(\bar{\beta}(\log b)' - a\bar{\beta} - \bar{\beta}').$$
 (48)

In this case, its second flow (21) takes the following form (see eq. (50) in [14])

$$\partial_{2}b = (-b'' + 2ba)' + 2b(\frac{\beta\beta}{b})', \quad \partial_{2}\bar{\beta} = (\bar{\beta}' + 2(a\bar{\beta} - \bar{\beta}(\log b)'))', \\ \partial_{2}a = (a' + a^{2} + 2(b + \beta(\frac{\bar{\beta}}{b})' + a\frac{\beta\bar{\beta}}{b}))', \quad \partial_{2}\beta = (-\beta' + 2a\beta)',$$
(49)

which is related to the second flow of the a=4, N=2 super-KdV hierarchy [15].

The discrete symmetry (45) can equivalently be rewritten in terms of the N=2 superfields entering into the N=4 super-KdV hierarchy. It reads<sup>2</sup>

$$\mathcal{F}_{i+1}\overline{\mathcal{F}}_{i+1} = \Phi_i + D\overline{D}\mathcal{J}_i - (\log \mathcal{F}_i)'' + \frac{1}{\Phi_i}(\overline{D}\mathcal{J}_i + \left(\frac{\overline{D}\mathcal{F}_i}{\mathcal{F}_i}\right)')(D\mathcal{J}_i' - D\mathcal{J}_i(\log \mathcal{F}_i)' - \mathcal{F}_i D\overline{\mathcal{F}_i}),$$

$$\mathcal{F}_i \overline{\mathcal{F}}_{i+1} = \Phi_i, \qquad \mathcal{J}_{i+1} = \mathcal{J}_i + (\log \Phi_i)', \qquad \Phi_i \equiv \overline{D} D \mathcal{J}_i + \mathcal{F}_i \overline{\mathcal{F}}_i + \frac{D \mathcal{J}_i D \mathcal{F}_i}{\mathcal{F}_i}.$$
(50)

The discrete symmetries (45) and (50) are useful when constructing the solutions of the N=4 supersymmetric KdV and (1,1)-GNLS hierarchies. For example, if the set  $\{\mathcal{J}_i, \mathcal{F}_i, \overline{\mathcal{F}}_i\}$  is a solution of the N=4 super-KdV hierarchy, then the set  $\{\mathcal{J}_{i+1}, \mathcal{F}_{i+1}, \overline{\mathcal{F}}_{i+1}\}$  is a solution of this hierarchy as well. Let us note that eqs. (45) and (50) set aside boundary conditions. Therefore, via eqs. (30) and (42) one can obtain a solution of the N=4 super-KdV hierarchy in terms of solutions of the generalized N=4 supersymmetric TL hierarchy for different boundary conditions including periodic ones [1, 16].

## 5 Conclusion

In this paper, we demonstrated that the generalized N=4 supersymmetric TL hierarchy contains the N=4 supersymmetric KdV one. We used the first flow of the generalized N=4 supersymmetric TL hierarchy in order to express all the lattice fields in terms of the fields defined in the

 $<sup>^{2}</sup>$ In [4], the N=4 superfield form of the Darboux-Backlund symmetries of the N=4 super-KdV-Toda hierarchy, which are related to eqs. (45) and (50), was presented.

same lattice node and rewrote all its flows in the form of differential equations. In such a way, we reproduced the component form of the N=4 KdV hierarchy as well as its supersymmetric transformations and conservation laws. Finally, we obtained two different N=2 superfield forms of the generalized N=4 supersymmetric TL equation which are helpful when solving the N=4 super-KdV and (1,1)-GNLS hierarchies.

In conclusion, we would like to note that the one-dimensional generalized N=4 supersymmetric TL hierarchy is a particular case of a wide class of hierarchies [17] defined by the Lax operators

$$(L)_{i,j} = \sum_{k=-2}^{2n} u_{k,i} \delta_{i,j+k}, \quad u_{-2,i} = 1, \quad n > 0.$$
 (51)

These hierarchies possess the N=2 supersymmetry and it would be interesting to investigate which N=2 supersymmetric differential hierarchies can be reproduced from them in our approach.

Another problem of special interest is to consider the Lax operator (2) with bosonic and fermionic fields replaced by square  $k \times k$  matrices with bosonic and fermionic entries, respestively. In our approach, such a Lax operator gives rise to the matrix hierarchy which is N=4 supersymmetric and under reduction constraints, when all the off-diagonal fields are equal to zero, splits into k independent N=4 super-KdV hierarchies. One can expect that such matrix hierarchy can throw light on the longstanding problem of constructing the N=4 super-KdV hierarchy (if any) with the N=4 O(4) ("large") superconformal algebra as the second Hamiltonian structure.

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