Weakly nonassociative algebras, Riccati and KP hierarchies

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Abstract

It has recently been observed that certain nonassociative algebras (called 'weakly nonassociative', WNA) determine, via a universal hierarchy of ordinary differential equations, solutions of the KP hierarchy with dependent variable in an associative subalgebra (the middle nucleus). We recall central results and consider a class of WNA algebras for which the hierarchy of ODEs reduces to a matrix Riccati hierarchy, which can be easily solved. The resulting solutions of a matrix KP hierarchy determine, under a 'rank one condition', solutions of the scalar KP hierarchy. We extend these results to the discrete KP hierarchy. Moreover, we build a bridge from the WNA framework to the Gelfand-Dickey formulation of the KP hierarchy.

1 Introduction

The Kadomtsev-Petviashvili (KP) equation is an extension of the famous Korteweg-deVries (KdV) equation to 2+1 dimensions. It first appeared in a stability analysis of KdV solitons [1, 2]. In particular, it describes nonlinear fluid surface waves in a certain approximation and explains to some extent the formation of network patterns formed by line wave segments on a water surface [2]. It is 'integrable' in several respects, in particular in the sense of the inverse scattering method. Various remarkable properties have been discovered that allow to access (subsets of) its solutions in different ways, see in particular [3–5]. Apart from its direct relevance in physics, the KP equation and its hierarchy (see [5, 6], for example) is deeply related to the theory of Riemann surfaces (Riemann-Schottky problem, see [7] for a review). Some time ago, this stimulated discussions concerning the role of KP in string theory (see [8–11], for example). Later the Gelfand-Dickey hierarchies, of which the KdV hierarchy is the simplest and which are reductions of the KP hierarchy, made their appearance in matrix models, first in a model of two-dimensional quantum gravity (see [12, 13] and references therein). This led to important developments in algebraic geometry (see [14], for example). Of course, what we mentioned here by far does not exhaust what is known about KP and there is probably even much more in the world of mathematics and physics linked to the KP equation and its descendants that still waits to be uncovered.

In fact, an apparently completely different appearance of the KP hierarchy has been observed in [15]. On a freely generated 'weakly nonassociative' (WNA) algebra (see section 2) there is a family of commuting derivations[1] that satisfy identities which are in correspondence with the equations of the KP hierarchy (with dependent variable in a noncommutative associative subalgebra). As a consequence, there is a hierarchy of ordinary differential equations (ODEs) on this WNA algebra that implies the KP hierarchy. More generally, this holds for *any* WNA algebra. In this way WNA algebras determine classes of solutions of the KP hierarchy.

[1] Families of commuting derivations on certain algebras also appeared in [16, 17], for example. In fact, the ideas underlying the work in [15] grew out of our work in [18] which has some algebraic overlap with [16].

In section 2 we recall central results of [15] and present a new result in proposition 1. Section 3 applies the WNA approach to derive a matrix Riccati[2] hierarchy, the solutions of which are solutions of the corresponding matrix KP hierarchy (which under certain conditions determines solutions of the scalar KP hierarchy). In section 4 we extend these results to the discrete KP hierarchy [35–39]. Furthermore, in section 5 we show how the Gelfand-Dickey formulation [5] of the KP hierarchy (with dependent variable in any associative algebra) emerges in the WNA framework. Section 6 contains some conclusions.

2 Nonassociativity and KP

In [15] we called an algebra $(\mathbb{A}, \circ)$ (over a commutative ring) *weakly nonassociative (WNA)* if

$$(a, b \circ c, d) = 0 \qquad \forall a, b, c, d \in \mathbb{A}\,, \tag{2.1}$$

where $(a, b, c) := (a \circ b) \circ c - a \circ (b \circ c)$ is the associator in $\mathbb{A}$. The *middle nucleus* of $\mathbb{A}$ (see e.g. [40]),

$$\mathbb{A}' := \{b \in \mathbb{A} \mid (a, b, c) = 0 \ \ \forall a, c \in \mathbb{A}\}\,, \tag{2.2}$$

is an *associative* subalgebra and a two-sided ideal. We fix $f \in \mathbb{A}$, $f \notin \mathbb{A}'$, and define $a \circ_1 b := a \circ b$,

$$a \circ_{n+1} b := a \circ (f \circ_n b) - (a \circ f) \circ_n b\,, \qquad n = 1, 2, \ldots\ . \tag{2.3}$$

As a consequence of (2.1), these products only depend on the equivalence class $[f]$ of f in $\mathbb{A}/\mathbb{A}'$. The subalgebra $\mathbb{A}(f)$, generated by f in the WNA algebra $\mathbb{A}$, is called *δ-compatible* if, for each $n \in \mathbb{N}$,

$$\delta_n(f) := f \circ_n f \tag{2.4}$$

extends to a *derivation* of $\mathbb{A}(f)$. In the following we recall some results from [15].

Theorem 1 *Let $\mathbb{A}(f)$ be δ-compatible. The derivations δ_n commute on $\mathbb{A}(f)$ and satisfy identities that are in correspondence via $\delta_n \mapsto \partial_{t_n}$ (the partial derivative operator with respect to a variable t_n) with the equations of the potential Kadomtsev-Petviashvili (pKP) hierarchy with dependent variable in $\mathbb{A}'$.* □

This is a central observation in [15] with the following immediate consequence.

Theorem 2 *Let $\mathbb{A}$ be any WNA algebra over the ring of complex functions of independent variables $t_1, t_2, \ldots$. If $f \in \mathbb{A}$ solves the hierarchy of ODEs*[3]

$$f_{t_n} := \partial_{t_n}(f) = f \circ_n f\,, \qquad n = 1, 2, \ldots\,, \tag{2.5}$$

then $-f_{t_1}$ lies in $\mathbb{A}'$ and solves the KP hierarchy with dependent variable in $\mathbb{A}'$. □

Corollary 1 *If there is a constant $\nu \in \mathbb{A}$, $\nu \notin \mathbb{A}'$, with $[\nu] = [f] \in \mathbb{A}/\mathbb{A}'$, then, under the assumptions of theorem 2,*

$$\phi := \nu - f \in \mathbb{A}' \tag{2.6}$$

[2] Besides their appearance in control and systems theory, matrix Riccati equations (see [19–21], for example) frequently showed up in the context of integrable systems, see in particular [22–34].

[3] f has to be differentiable, of course, which requires a corresponding (e.g. Banach space) structure on $\mathbb{A}$. The flows given by (2.5) indeed commute [15]. Furthermore, (2.5) implies δ-compatibility of the algebra $\mathbb{A}(f)$ generated by f in $\mathbb{A}$ over $\mathbb{C}$ [15].

solves the potential KP (pKP) hierarchy[4]

$$\sum_{i,j,k=1}^{3} \varepsilon_{ijk} \, \left(\lambda_i^{-1}(\phi_{[\lambda_i]} - \phi) + \phi \circ \phi_{[\lambda_i]}\right)_{[\lambda_k]} = 0 \,, \tag{2.7}$$

where ε_{ijk} *is totally antisymmetric with* $\varepsilon_{123} = 1$, λ_i, $i = 1,2,3$, *are indeterminates, and* $\phi_{\pm[\lambda]}(\mathbf{t}) := \phi(\mathbf{t} \pm [\lambda])$, *where* $\mathbf{t} = (t_1, t_2, \ldots)$ *and* $[\lambda] := (\lambda, \lambda^2/2, \lambda^3/3, \ldots)$. □

Remark 1. If $C \in \mathbb{A}'$ is constant, then $f = \nu' - (\phi + C)$ with constant $\nu' := \nu + C$ satisfying $[\nu'] = [\nu] = [f]$. Hence, with ϕ also $\phi + C$ is a solution of the pKP hierarchy. This can also be checked directly using (2.7), of course. □

The next result will be used in section 4.

Proposition 1 *Suppose* f *and* f' *solve (2.5) and* $[f] = [f']$ *in a WNA algebra* $\mathbb{A}$*. The equation*

$$f' \circ f = \alpha \, (f' - f) \tag{2.8}$$

is then preserved for all $\alpha \in \mathbb{C}$.

Proof:

$$\begin{aligned} (f' \circ f)_{t_n} &= f'_{t_n} \circ f + f' \circ f_{t_n} = (f' \circ_n f') \circ f + f' \circ (f \circ_n f) \\ &= (f' \circ_n f') \circ f - f' \circ_n (f' \circ f) + \alpha \, f' \circ_n (f' - f) \\ &\quad + f' \circ (f \circ_n f) - (f' \circ f) \circ_n f + \alpha \, (f' - f) \circ_n f \\ &= -f' \circ_{n+1} f + f' \circ_{n+1} f + \alpha \, (f' \circ_n f' - f \circ_n f) = \alpha \, (f' - f)_{t_n} \,. \end{aligned}$$

In the third step we have added terms that vanish as a consequence of (2.8). Then we used (3.11) in [15] (together with the fact that the products $\circ_n$ only depend on the equivalence class $[f] = [f'] \in \mathbb{A}/\mathbb{A}'$), and also (2.3), to combine pairs of terms into products of one degree higher. □

Remark 2. In functional form, (2.5) can be expressed (e.g. with the help of results in [15]) as

$$\lambda^{-1}(f - f_{-[\lambda]}) - f_{-[\lambda]} \circ f = 0 \,. \tag{2.9}$$

Setting $f' = f_{-[\lambda]}$ (which also solves (2.9) if f solves it), this takes the form (2.8) with $\alpha = -\lambda^{-1}$. □

In order to apply the above results, we need examples of WNA algebras. For our purposes, it is sufficient to recall from [15] that any WNA algebra with $\dim(\mathbb{A}/\mathbb{A}') = 1$ is isomorphic to one determined by the following data:

(1) an associative algebra $\mathcal{A}$ (e.g. any matrix algebra)
(2) a fixed element $g \in \mathcal{A}$
(3) linear maps $\mathcal{L}, \mathcal{R} \, : \, \mathcal{A} \to \mathcal{A}$ such that

$$[\mathcal{L}, \mathcal{R}] = 0 \,, \qquad \mathcal{L}(a \circ b) = \mathcal{L}(a) \circ b \,, \qquad \mathcal{R}(a \circ b) = a \circ \mathcal{R}(b) \,. \tag{2.10}$$

Augmenting $\mathcal{A}$ with an element f such that

$$f \circ f := g \,, \qquad f \circ a := \mathcal{L}(a) \,, \qquad a \circ f := \mathcal{R}(a) \,, \tag{2.11}$$

leads to a WNA algebra $\mathbb{A}$ with $\mathbb{A}' = \mathcal{A}$, provided that the following condition holds,

$$\exists \, a, b \in \mathcal{A} \, : \, \mathcal{R}(a) \circ b \neq a \circ \mathcal{L}(b) \,. \tag{2.12}$$

This guarantees that the augmented algebra is *not* associative. Particular examples of $\mathcal{L}$ and $\mathcal{R}$ are given by multiplication from left, respectively right, by fixed elements of $\mathcal{A}$ (see also the next section).

[4]This functional representation of the potential KP hierarchy appeared in [41, 42]. See also [15, 26] for equivalent formulae.

3 A class of WNA algebras and a matrix Riccati hierarchy

Let $\mathcal{M}(M,N)$ be the vector space of complex $M \times N$ matrices, depending smoothly on independent real variables $t_1, t_2, \ldots$, and let S, L, R, Q be constant matrices of dimensions $M \times N$, $M \times M$, $N \times N$ and $N \times M$, respectively. Augmenting with a constant element ν and setting[5]

$$\nu \circ \nu = -S\,, \qquad \nu \circ A = LA\,, \qquad A \circ \nu = -AR\,, \qquad A \circ B = AQB\,, \tag{3.1}$$

for all $A, B \in \mathcal{M}(M,N)$, we obtain a WNA algebra $(\mathbb{A}, \circ)$. The condition (2.12) requires

$$RQ \neq QL\,. \tag{3.2}$$

For the products $\circ_n$, $n > 1$, we have the following result.

Proposition 2

$$\nu \circ_n \nu = -S_n\,, \qquad \nu \circ_n A = L_n A\,, \qquad A \circ_n \nu = -AR_n\,, \qquad A \circ_n B = AQ_nB\,, \tag{3.3}$$

where

$$\begin{pmatrix} R_n & Q_n \\ S_n & L_n \end{pmatrix} = H^n \qquad \textit{with} \quad H := \begin{pmatrix} R & Q \\ S & L \end{pmatrix}\,. \tag{3.4}$$

Proof: Using the definition (2.3), one proves by induction that

$$S_{n+1} = LS_n + SR_n\,, \qquad L_{n+1} = LL_n + SQ_n\,, \quad R_{n+1} = QS_n + RR_n\,, \qquad Q_{n+1} = QL_n + RQ_n\,,$$

for $n = 1, 2, \ldots$, where $S_1 = S$, $L_1 = L$, $R_1 = R$, $Q_1 = Q$. This can be written as

$$\begin{pmatrix} R_{n+1} & Q_{n+1} \\ S_{n+1} & L_{n+1} \end{pmatrix} = H \begin{pmatrix} R_n & Q_n \\ S_n & L_n \end{pmatrix}\,,$$

which implies (3.4). □

Using (2.6) and (3.3) in (2.5), leads to the matrix Riccati equations[6]

$$\phi_{t_n} = S_n + L_n\phi - \phi R_n - \phi Q_n \phi\,, \qquad n = 1, 2, \ldots\,. \tag{3.5}$$

Solutions of (3.5) are obtained in a well-known way (see [21, 32], for example) via

$$\phi = YX^{-1} \tag{3.6}$$

from the linear system

$$Z_{t_n} = H^n Z\,, \qquad Z = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{3.7}$$

with an $N \times N$ matrix X and an $M \times N$ matrix Y, provided X is invertible. This system is solved by

$$Z(\mathbf{t}) = e^{\xi(H)} Z_0 \qquad \text{where} \quad \xi(H) := \sum_{n \geq 1} t_n H^n\,. \tag{3.8}$$

[5] Using (2.6), in terms of f this yields relations of the form (2.11).

[6] The corresponding functional form is $\lambda^{-1}(\phi_{[\lambda]} - \phi) + \phi Q \phi_{[\lambda]} = S + L\phi_{[\lambda]} - \phi R$, which is easily seen to imply (2.7), see also [43]. The appendix provides a FORM program [44, 45] which independently verifies that any solution of (3.5), reduced to $n = 1, 2, 3$, indeed solves the matrix pKP equation in $(\mathcal{M}(M,N), \circ)$.

If Q has rank 1, then

$$\varphi := \mathrm{tr}(Q\phi) \tag{3.9}$$

defines a homomorphism from $(\mathcal{M}(M,N),\circ)$ into the scalars (with the ordinary product of functions). Hence, if ϕ solves the pKP hierarchy in $(\mathcal{M}(M,N),\circ)$, then φ solves the scalar pKP hierarchy.[7] More generally, if $Q = VU^T$ with V, U of dimensions $N \times r$, respectively $M \times r$, then $U^T \phi V$ solves the $r \times r$-matrix KP hierarchy.

$GL(N+M,\mathbb{C})$ acts on the space of all $(N+M)\times(N+M)$ matrices H by similarity transformations. In a given orbit this allows to choose for H some 'normal form', for which we can evaluate (3.8) and then elaborate the effect of $GL(N+M,\mathbb{C})$ transformations (see also remark 3 below) on the corresponding solution of the pKP hierarchy, with the respective Q given by the normal form of H. By a similarity transformation we can always achieve that $Q = 0$ and the problem of solving the pKP hierarchy (with some non-zero Q) can thus in principle be reduced to solving its linear part. Alternatively, we can always achieve that $S = 0$ and the next two examples take this route.

Example 1. If $S = 0$, we can in general not achieve that also $Q = 0$. In fact, the matrices

$$H = \begin{pmatrix} R & Q \\ 0 & L \end{pmatrix} \quad \text{and} \quad H_0 := \begin{pmatrix} R & 0 \\ 0 & L \end{pmatrix} \tag{3.10}$$

are similar (i.e. related by a similarity transformation) if and only if the matrix equation $Q = RK - KL$ has an $N \times M$ matrix solution K [56–60], and then

$$H = \mathcal{T}\, H_0\, \mathcal{T}^{-1}\,, \qquad \mathcal{T} = \begin{pmatrix} I_N & -K \\ 0 & I_M \end{pmatrix}\,. \tag{3.11}$$

It follows that

$$H^n = \mathcal{T}\, H_0^n\, \mathcal{T}^{-1} = \begin{pmatrix} R^n & R^n K - K L^n \\ 0 & L^n \end{pmatrix} \tag{3.12}$$

and thus

$$e^{\xi(H)} = \begin{pmatrix} e^{\xi(R)} & e^{\xi(R)} K - K e^{\xi(L)} \\ 0 & e^{\xi(L)} \end{pmatrix}\,. \tag{3.13}$$

If (3.2) holds, we obtain the following solution of the matrix pKP hierarchy in $(\mathcal{M}(M,N),\circ)$,

$$\phi = e^{\xi(L)} \phi_0 \, (I_N + K\phi_0 - e^{-\xi(R)} K e^{\xi(L)} \phi_0)^{-1} e^{-\xi(R)}\,, \tag{3.14}$$

where $\phi_0 = Y_0 X_0^{-1}$. This in turn leads to

$$\begin{aligned} \varphi &= \mathrm{tr}\Big(e^{-\xi(R)} (RK - KL) e^{\xi(L)} \phi_0 \, (I_N + K\phi_0 - e^{-\xi(R)} K e^{\xi(L)} \phi_0)^{-1} \Big) \\ &= \mathrm{tr}\Big(\log(I_N + K\phi_0 - e^{-\xi(R)} K e^{\xi(L)} \phi_0) \Big)_{t_1} \\ &= (\log \tau)_{t_1}\,, \qquad \tau := \det(I_N + K\phi_0 - e^{-\xi(R)} K e^{\xi(L)} \phi_0)\,. \end{aligned} \tag{3.15}$$

If $\mathrm{rank}(Q) = 1$, then φ solves the scalar pKP hierarchy. Besides (3.2) and this rank condition, further conditions will have to be imposed on the (otherwise arbitrary) matrices R, K, L and ϕ_0 to achieve that φ

[7] For related results and other perspectives on the rank one condition, see [46] and the references cited there. The idea to look for (simple) solutions of matrix and more generally operator versions of an 'integrable' equation, and to generate from it (complicated) solutions of the scalar equation by use of a suitable map, already appeared in [47] (see also [48–55]).

is a *real* and *regular* solution. See [61], and references cited there, for classes of solutions obtained from an equivalent formula or restrictions of it. This includes multi-solitons and soliton resonances (KP-II), and lump solutions (passing to KP-I via $t_{2n} \mapsto i\,t_{2n}$ and performing suitable limits of parameters).

Example 2. If $M = N$ and $S = 0$, let us consider

$$H = \mathcal{T}\, H_0\, \mathcal{T}^{-1}\,, \qquad H_0 = \begin{pmatrix} L & I \\ 0 & L \end{pmatrix}\,, \qquad \mathcal{T} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}\,, \tag{3.16}$$

with $I = I_N$ and a constant $N \times N$ matrix K. As a consequence,

$$Q = I + [L, K]\;. \tag{3.17}$$

We note that H_0 is *not* similar to $\mathrm{diag}(L, L)$ [60]. Now we obtain

$$H^n = \mathcal{T}\, H_0^n\, \mathcal{T}^{-1} = \begin{pmatrix} L^n & n\,L^{n-1} + [L^n, K] \\ 0 & L^n \end{pmatrix} \tag{3.18}$$

and furthermore

$$e^{\xi(H)} = \begin{pmatrix} e^{\xi(L)} & \sum_{n\geq 1} n\,t_n\,L^{n-1} e^{\xi(L)} + [e^{\xi(L)}, K] \\ 0 & e^{\xi(L)} \end{pmatrix}\;. \tag{3.19}$$

If $[L, [L, K]] \neq 0$ (which is condition (3.2)), we obtain the solution

$$\phi = e^{\xi(L)} \phi_0\, (I + K\phi_0 + F)^{-1} e^{-\xi(L)} \tag{3.20}$$

of the matrix pKP hierarchy in $(\mathcal{M}(N, N), \circ)$, where

$$F := \Big(\sum_{n\geq 1} n\,t_n\,L^{n-1} - e^{-\xi(L)} K e^{\xi(L)}\Big)\phi_0\;. \tag{3.21}$$

Furthermore, using $F_{t_1} = e^{-\xi(L)}(I + [L, K])e^{\xi(L)}\phi_0$, we find

$$\varphi = \mathrm{tr}((I + [L, K])\phi) = \mathrm{tr}(F_{t_1}(I + K\phi_0 + F)^{-1}) = (\mathrm{tr}\log(I + K\phi_0 + F))_{t_1} \tag{3.22}$$

and thus

$$\varphi = (\log\tau)_{t_1}\,, \qquad \tau := \det\Big(I + K\phi_0 + (\sum_{n\geq 1} n\,t_n\,L^{n-1} - e^{-\xi(L)} K e^{\xi(L)})\phi_0\Big)\;. \tag{3.23}$$

If $\mathrm{rank}(I + [L, K]) = 1$ (see also [46, 62, 63] for appearances of this condition), then φ solves the scalar pKP hierarchy. Assuming that ϕ_0 is invertible, we can rewrite τ as follows,

$$\tau = \det\Big(e^{\xi(L)}(\phi_0^{-1} + K)e^{-\xi(L)} + \sum_{n\geq 1} n\,t_n\,L^{n-1} - K\Big) \tag{3.24}$$

(dropping a factor $\det(\phi_0)$). This simplifies considerably if we set $\phi_0^{-1} = -K$.[8] Choosing moreover

$$L_{ij} = -(q_i - q_j)^{-1} \quad i \neq j\,, \quad L_{ii} = -p_i\,, \quad K = \mathrm{diag}(q_1, \ldots, q_N)\,, \tag{3.25}$$

(3.24) reproduces a polynomial (in any finite number of the t_n) tau function associated with Calogero-Moser systems [46, 62, 63]. Alternatively, we may choose

$$L = \mathrm{diag}(q_1, \ldots, q_N)\,, \quad K_{ij} = (q_i - q_j)^{-1} \quad i \neq j\,, \quad K_{ii} = p_i\;. \tag{3.26}$$

[8] Note that in this case $\phi = (\sum_{n\geq 1} n\,t_n\,L^{n-1} - K)^{-1}$, which is rational in any finite number of the variables t_n.

The corresponding solutions of the KP-I hierarchy ($t_{2n} \mapsto i\,t_{2n}$) include the rational soliton solutions ('lumps') originally obtained in [64]. In particular, $N=2$ and $q_2=-\bar{q}_1$, $p_2=\bar{p}_1$ (where the bar means complex conjugation), yields the single lump solution given by

$$\tau = |p_1+\xi'(q_1)|^2 + \frac{1}{4\Re(q_1)^2} \quad \text{where} \quad \xi'(q) := \sum_{n\geq 1} n\, t_n\, q^{n-1}\Big|_{\{t_{2k}\mapsto i\,t_{2k}\,,\,k=1,2,\ldots\}} . \tag{3.27}$$

Example 3. Let $M=N$ and $L=S\pi_-$, $R=\pi_+S$, $Q=\pi_+S\pi_-$, with constant $N\times N$ matrices π_+,π_- subject to $\pi_++\pi_-=I$. The matrix H can then be written as

$$H = \begin{pmatrix} \pi_+ \\ I \end{pmatrix} S \begin{pmatrix} I & \pi_- \end{pmatrix} , \tag{3.28}$$

which lets us easily calculate

$$H^n = \begin{pmatrix} \pi_+S^n & \pi_+S^n\pi_- \\ S^n & S^n\pi_- \end{pmatrix} . \tag{3.29}$$

As a consequence, we obtain

$$\phi = (-C_+ + e^{\xi(S)}C_-)(\pi_-C_+ + \pi_+e^{\xi(S)}C_-)^{-1} , \tag{3.30}$$

where $C_\pm := I \mp \pi_\pm\phi_0$. This solves the matrix pKP hierarchy in $\mathcal{M}(M,N)$ with the product $A\circ B = A\pi_+S\pi_-B$ if (3.2) holds, which is $\pi_+S(\pi_+-\pi_-)S\pi_- \neq 0$. If furthermore $\mathrm{rank}(\pi_+S\pi_-)=1$, then

$$\varphi = \mathrm{tr}(Q\phi) = -\mathrm{tr}(\pi_+S) + (\log\tau)_{t_1} , \qquad \tau = \det(\pi_-C_+ + \pi_+e^{\xi(S)}C_-) \tag{3.31}$$

solves the scalar pKP hierarchy (see also [43]). We will meet the basic structure underlying this example again in section 5.

Remark 3. A $GL(N+M,\mathbb{C})$ matrix

$$\mathcal{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.32}$$

can be decomposed as follows,

$$\mathcal{T} = \begin{pmatrix} I_N & BD^{-1} \\ 0 & I_M \end{pmatrix} \begin{pmatrix} S_D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_N & 0 \\ D^{-1}C & I_M \end{pmatrix} , \tag{3.33}$$

if D and its Schur complement $S_D = A - BD^{-1}C$ are both invertible. Let us see what effect the three parts of $\mathcal{T}$ induce on ϕ when acting on Z.

(1) Writing $P = D^{-1}C$, the first transformation leads to $\phi \mapsto \phi + P$, a shift by the constant matrix P.

(2) The second transformation amounts to $\phi \mapsto D\phi S_D^{-1}$ (where ϕ is now the result of the previous transformation).

(3) Writing $K=-BD^{-1}$, the last transformation is $\phi \mapsto \phi\,(I_N - K\phi)^{-1}$. □

4 WNA algebras and solutions of the discrete KP hierarchy

The potential discrete KP (pDKP) hierarchy in an associative algebra $(\mathcal{A},\circ)$ can be expressed in functional form as follows,[9]

$$\Omega(\lambda)^+ - \Omega(\lambda)_{-[\mu]} = \Omega(\mu)^+ - \Omega(\mu)_{-[\lambda]} , \tag{4.1}$$

[9] This functional representation of the pDKP hierarchy is equivalent to (3.32) in [39].

where λ, μ are indeterminates,

$$\Omega(\lambda) := \lambda^{-1}(\phi - \phi_{-[\lambda]}) - (\phi^+ - \phi_{-[\lambda]}) \circ \phi \,, \tag{4.2}$$

and $\phi = (\phi_k)_{k \in \mathbb{Z}}$, $\phi_k^+ := \phi_{k+1}$. The pDKP hierarchy implies that each component ϕ_k, $k \in \mathbb{Z}$, satisfies the pKP hierarchy and its remaining content is a special pKP Bäcklund transformation (BT) acting between neighbouring sites on the linear lattice labeled by k [35, 39]. This suggests a way to extend the method of section 3 to construct exact solutions of the pDKP hierarchy. What is needed is a suitable extension of (2.5) that accounts for the BT and this is offered by proposition 1.

Theorem 3 *Let $\mathbb{A}$ be a WNA algebra with a constant element $\nu \in \mathbb{A}$, $\nu \notin \mathbb{A}'$. Any solution*

$$f = (\nu - \phi_k)_{k \in \mathbb{Z}} \,, \tag{4.3}$$

of the hierarchy (2.5) together with the compatible constraint[10]

$$f^+ \circ f = 0 \tag{4.4}$$

yields a solution $\phi = (\phi_k)_{k \in \mathbb{Z}}$ of the pDKP hierarchy in $\mathbb{A}'$.

Proof: Since $[f^+] = [f]$, the compatibility follows by setting $f' = f^+$ and $\alpha = 0$ in proposition 1. Using $f_{t_1} = f \circ f$, we rewrite (2.9) as

$$\lambda^{-1}(f - f_{-[\lambda]}) + (f - f_{-[\lambda]}) \circ f - f_{t_1} = 0 \,.$$

Inserting $f = \nu - \phi$, this takes the form

$$\lambda^{-1}(\phi - \phi_{-[\lambda]}) - \phi_{t_1} - (\phi - \phi_{-[\lambda]}) \circ \phi = \theta - \theta_{-[\lambda]}$$

with $\theta := -\phi \circ \nu$. Next we use (4.4) and $f_{t_1} = f \circ f$ to obtain $(f^+ - f) \circ f + f_{t_1} = 0$, which is

$$\phi_{t_1} - (\phi^+ - \phi) \circ \phi = \theta^+ - \theta \,.$$

Together with the previous equation, this leads to

$$\lambda^{-1}(\phi - \phi_{-[\lambda]}) - (\phi^+ - \phi_{-[\lambda]}) \circ \phi = \theta^+ - \theta_{-[\lambda]}$$

(which is actually equivalent to the last two equations), so that

$$\Omega(\lambda) = \theta^+ - \theta_{-[\lambda]} \,.$$

This is easily seen to solve (4.1). □

Let us choose the WNA algebra of section 3.[11] Evaluation of (2.5) leads to the matrix Riccati hierarchy (3.5), and (4.4) with $f^+ = \nu + C - \phi^+$ becomes

$$S + CR + (L + CQ)\phi - \phi^+ R - \phi^+ Q \phi = 0 \,, \tag{4.5}$$

which can be rewritten as

$$\phi^+ = (S + L\phi)(R + Q\phi)^{-1} + C = Y^+ \, (X^+)^{-1} \tag{4.6}$$

[10] Note that (4.4) implies $f^{n+} \circ_n f = 0$, where $f_k^{n+} := f_{k+n}$. This follows by induction from $f^{(n+1)+} \circ_{n+1} f = f^{(n+1)+} \circ (f^{n+} \circ_n f) - (f^{(n+1)+} \circ f^{n+}) \circ_n f = f^{(n+1)+} \circ (f^{n+} \circ_n f) - (f^+ \circ f)^{n+} \circ_n f$, where we used (2.3) and $[f^{n+}] = [f]$ in the first step.

[11] Since there is only a single element ν, the matrices L, R, S do not depend on the discrete variable k.

(assuming that the inverse matrices exist), where X^+, Y^+ are the components of

$$Z^+ = TH\, Z = TH\, e^{\xi(H)} Z^{(0)} \,, \tag{4.7}$$

with Z, H, T taken from section 3. Deviating from the notation of section 3, we write $Z^{(0)}$ for the constant vector, since Z_0 should now denote the component of Z at the lattice site 0. In order that (4.7) defines a pDKP solution on the whole lattice, we need H invertible. Since the matrix C, and thus also T, may depend on the lattice site k, solutions of (4.1) are determined by

$$Z_k = T_k H T_{k-1} H \cdots T_1 H Z_0 \,, \quad Z_{-k} = (T_{-k}H)^{-1}(T_{-k+1}H)^{-1} \cdots (T_{-1}H)^{-1} Z_0 \,, \qquad k \in \mathbb{N} \,. \tag{4.8}$$

This corresponds to a sequence of transformations applied to the matrix pKP solution ϕ_0 determined by Z_0, which generate new pKP solutions (cf. [35]). ϕ_1 is then given by (4.6) in terms of ϕ_0, and

$$\begin{aligned} \phi_2 \;&=\; [LS + SR + LC_1R + (L^2 + SQ + LC_1Q)\phi_0] \\ &\quad \times [R^2 + QS + QC_1R + (QL + RQ + QC_1Q)\phi_0]^{-1} + C_2 \end{aligned} \tag{4.9}$$

shows that the action of the T_k becomes considerably more involved for $k > 1$. In the special case $T_k = I_{N+M}$ (so that $C_k = 0$), we have

$$Z_k = e^{\xi(H)}(H^k Z_0^{(0)}) \qquad k \in \mathbb{Z} \,. \tag{4.10}$$

If $X_k^{(0)}, Y_k^{(0)}$ are the components of the vector $H^k Z_0^{(0)}$, the lattice component ϕ_k of the pDKP solution determined in this way is therefore just given by the pKP solution of section 3 with initial data (at $\mathbf{t} = 0$)

$$\phi_k^{(0)} = Y_k^{(0)}(X_k^{(0)})^{-1} = L^k \phi_0^{(0)}\, [R^k + (R^k K - K L^k)\phi_0^{(0)}]^{-1} \,. \tag{4.11}$$

With the restrictions of example 1 in section 3, assuming that L and R are invertible (so that H is invertible), the corresponding solution of the matrix pDKP hierarchy (in the matrix algebra with product $A \circ B = A(RK - KL)B$) is

$$\phi_k = e^{\xi(L)} L^k \phi_0^{(0)} [R^k (I_N + K\phi_0^{(0)}) - e^{-\xi(R)} K e^{\xi(L)} L^k \phi_0^{(0)}]^{-1} e^{-\xi(R)} \,, \qquad k \in \mathbb{Z} \,, \tag{4.12}$$

which leads to

$$\varphi_k = (\log \tau_k)_{t_1} \qquad \text{with} \quad \tau_k = \det\Big(R^k (I_N + K\phi_0^{(0)}) - e^{-\xi(R)} K e^{\xi(L)} L^k \phi_0^{(0)} \Big) \qquad k \in \mathbb{Z} \,. \tag{4.13}$$

If $Q = RK - KL$ has rank 1, this is a solution of the scalar pDKP hierarchy.[12] As a special case, let us choose $M = N$, $L = \mathrm{diag}(p_1, \ldots, p_N)$, $R = \mathrm{diag}(q_1, \ldots, q_N)$, and K with entries $K_{ij} = (q_i - p_j)^{-1}$.[13] Then Q has rank 1 and we obtain N-soliton tau functions of the scalar discrete KP hierarchy. These can also be obtained via the Birkhoff decomposition method using appropriate initial data as in [65, 66].

With the assumptions made in example 2 of section 3, setting $\phi_0^{(0)} = -K^{-1}$, assuming that K and L are invertible, and choosing for T_k the identity, we find the matrix pDKP solution

$$\phi_k = \Big(\sum_{n \geq 1} n\, t_n\, L^{n-1} + k\, L^{-1} - K \Big)^{-1} \,, \qquad k \in \mathbb{Z} \,. \tag{4.14}$$

If $\mathrm{rank}(I_N + [L, K]) = 1$, this leads to the following solution of the scalar pDKP hierarchy,

$$\varphi_k = (\log \tau_k)_{t_1} \qquad \text{with} \quad \tau_k = \det\Big(\sum_{n \geq 1} n\, t_n\, L^{n-1} + k\, L^{-1} - K \Big) \,. \tag{4.15}$$

In example 3 of section 3, H is not invertible, so that (4.7) does not determine a pDKP solution.

---

12 Recall that $\varphi = \mathrm{tr}(Q\phi)$ (cf. 3.9) determines a homomorphism if Q has rank 1. As a consequence, if ϕ solves the matrix pDKP hierarchy (4.1), then φ solves the scalar pDKP hierarchy.

13 The condition (3.2) requires $q_i \neq p_j$ for all $i, j = 1, \ldots, N$.

5 From WNA to Gelfand-Dickey

Let $\mathcal{R}$ be the complex algebra of pseudo-differential operators [5]

$$\mathcal{V} = \sum_{i \ll \infty} v_i \, \partial^i \,, \tag{5.1}$$

with coefficients $v_i \in \mathfrak{A}$, where $\mathfrak{A}$ is the complex differential algebra of polynomials in (in general noncommuting) symbols $u_n^{(m)}$, $m = 0, 1, 2, \ldots$, $n = 2, 3, \ldots$, where $\partial(u_n^{(m)}) = u_n^{(m+1)}$ and $\partial(vw) = \partial(v)\, w + v\, \partial(w)$ for $v, w \in \mathfrak{A}$. We demand that $u_n^{(m)}$, $n = 2, 3, \ldots$, $m = 0, 1, 2, \ldots$, are algebraically independent in $\mathfrak{A}$, and we introduce the following linear operators on $\mathcal{R}$,

$$S(\mathcal{V}) := \mathfrak{L}\mathcal{V} \,, \qquad \pi_+(\mathcal{V}) := \mathcal{V}_{\geq 0} \,, \qquad \pi_-(\mathcal{V}) := \mathcal{V}_{<0} := \mathcal{V} - \mathcal{V}_{\geq 0} \,, \tag{5.2}$$

where $\mathcal{V}_{\geq 0}$ is the projection of a pseudo-differential operator $\mathcal{V}$ to its differential operator part, and

$$\mathfrak{L} = \partial + u_2 \, \partial^{-1} + u_3 \, \partial^{-2} + \cdots \,. \tag{5.3}$$

Let I denote the identity of $\mathcal{R}$ (which we identify with the identity in $\mathfrak{A}$), and let $\mathcal{O}$ be the subspace of linear operators on $\mathcal{R}$ spanned by S and elements of the form $S\pi_\pm S\pi_\pm \cdots \pi_\pm S$ (with any combination of signs). $\mathcal{O}$ becomes an algebra with the product given by

$$A \circ B := A\pi_+ S\pi_- B \,. \tag{5.4}$$

$(\mathcal{O}, \circ)$ is then generated by the elements $(S\pi_-)^m S\, (\pi_+ S)^n$, $m, n = 0, 1, \ldots$. Let us furthermore introduce $\mathcal{A} := \{v \in \mathfrak{A} \, : \, v = \mathrm{res}(A(I)), \; A \in \mathcal{O}\}$, where res takes the residue (the coefficient of ∂^{-1}) of a pseudo-differential operator. This is a subalgebra of $\mathfrak{A}$, since for $A, B \in \mathcal{O}$ we have

$$\mathrm{res}(A(I))\, \mathrm{res}(B(I)) = \mathrm{res}(A\pi_+ S\pi_- B(I)) \,, \tag{5.5}$$

so that the product of elements of $\mathcal{A}$ is again in $\mathcal{A}$. As a consequence of this relation (read from right to left), $\mathcal{A}$ is generated by the elements $\mathrm{res}((S\pi_-)^m S(\pi_+ S)^n(I))$, $m, n = 0, 1, \ldots$. Based on the following preparations, we will argue that $\mathcal{A}$ and $(\mathcal{O}, \circ)$ are actually isomorphic algebras.

Lemma 1 *For all $\mathcal{V} \in \mathcal{R}$,*

$$\mathrm{res}((S\pi_-)^m \mathcal{V}) = \mathrm{res}(\mathcal{D}_m \, \mathcal{V}) \,, \qquad m = 0, 1, \ldots \,, \tag{5.6}$$

where $\mathcal{D}_0 = I$ and $\{\mathcal{D}_m\}_{m=1}^\infty$ are the differential operators recursively determined by $\mathcal{D}_m = (\mathcal{D}_{m-1}\mathfrak{L})_{\geq 0}$.

Proof: We do the calculation for $m = 2$. This is easily generalized to arbitrary $m \in \mathbb{N}$.

$$\mathrm{res}((S\pi_-)^2 \, \mathcal{V}) = \mathrm{res}(\mathfrak{L}(\mathfrak{L}\mathcal{V}_{<0})_{<0}) = \mathrm{res}(\mathfrak{L}_{\geq 0}\mathfrak{L}\mathcal{V}_{<0}) = \mathrm{res}((\mathfrak{L}_{\geq 0}\mathfrak{L})_{\geq 0}\mathcal{V}) = \mathrm{res}(\mathcal{D}_2 \, \mathcal{V}) \,. \qquad \square$$

Proposition 3

$$\mathrm{res}((S\pi_-)^m S(\pi_+ S)^n(I)) = \sum_{k=0}^{m} \binom{m}{k} u^{(k)}_{m+n+2-k} + \textit{terms nonlinear in } u_k^{(j)} \,, \quad m, n = 0, 1, \ldots \tag{5.7}$$

Proof: According to the preceding lemma, we have

$$\mathrm{res}((S\pi_-)^m S(\pi_+ S)^n(I)) = \mathrm{res}(\mathcal{D}_m S(\pi_+ S)^n(I)) \ .$$

Next we note that $\mathcal{D}_m = \partial^m + D_m$, $(\pi_+ S)^n(I) = \partial^n + D'_n$ with differential operators D_m, D'_n (of degree smaller than m, respectively n) such that each of its summands contains factors from $\{u_k^{(j)}\}$ (so their coefficients are non-constant polynomials in the $u_k^{(j)}$). It follows that

$$\begin{aligned} \mathrm{res}((S\pi_-)^m S(\pi_+ S)^n(I)) &= \mathrm{res}((\partial^m + D_m)\mathfrak{L}_{<0}(\partial^n + D'_n)) \\ &= \mathrm{res}(\partial^m \mathfrak{L}_{<0}\partial^n) + \text{terms nonlinear in } u_k^{(j)} \ . \end{aligned}$$

It remains to evaluate

$$\begin{aligned} \mathrm{res}(\partial^m \mathfrak{L}_{<0}\partial^n) &= \sum_{j=1}^{\infty} \mathrm{res}(\partial^m u_{1+j}\partial^{n-j}) = \sum_{j=1}^{\infty} \mathrm{res}\Big(\sum_{k=0}^{m} \binom{m}{k} u_{1+j}^{(k)} \partial^{m+n-j-k}\Big) \\ &= \sum_{k=0}^{m} \binom{m}{k} u_{m+n+2-k}^{(k)} \ . \end{aligned}$$

□

According to the last proposition, the linear term with the highest derivative[14] in the residue of $(S\pi_-)^m S(\pi_+ S)^n(I)$ is given by $u_{n+2}^{(m)}$. We conclude that the monomials $(S\pi_-)^m S(\pi_+ S)^n$, $m, n = 0, 1, \ldots$, are algebraically independent in $(\mathcal{O}, \circ)$, since any algebraic relation among them would induce a corresponding algebraic relation in the set of $u_n^{(m)}$, but we assumed the $u_n^{(m)}$ to be algebraically independent. Together with (5.5), this implies that $\mathcal{A}$ and $(\mathcal{O}, \circ)$ are isomorphic algebras.

The last result allows us to introduce a WNA structure directly on $\mathcal{A}$ as follows.[15] Augmenting $\mathcal{A}$ with f such that, for $\mathcal{V}, \mathcal{W} \in \mathcal{O}(I)$,

$$\begin{aligned} f \circ f &:= -\mathrm{res}(\mathfrak{L}) \,, & f \circ \mathrm{res}(\mathcal{V}) &:= \mathrm{res}(\mathfrak{L}\mathcal{V}_{<0}) \,, \\ \mathrm{res}(\mathcal{V}) \circ f &:= -\mathrm{res}(\mathcal{V}_{<0}\mathfrak{L}) \,, & \mathrm{res}(\mathcal{V}) \circ \mathrm{res}(\mathcal{W}) &:= \mathrm{res}(\mathcal{V})\,\mathrm{res}(\mathcal{W}) \,, \end{aligned} \tag{5.8}$$

indeed defines a WNA algebra $\mathbb{A} = \mathbb{A}(f)$. The relations (5.8) are well-defined since $\mathrm{res}(A(I))$ uniquely determines $A \in \mathcal{O}$. By induction we obtain

$$\begin{aligned} f \circ_n f &= -\mathrm{res}(\mathfrak{L}^n) \,, & f \circ_n \mathrm{res}(\mathcal{V}) &= \mathrm{res}(\mathfrak{L}^n \mathcal{V}_{<0}) \,, \\ \mathrm{res}(\mathcal{V}) \circ_n f &= -\mathrm{res}(\mathcal{V}_{<0}\mathfrak{L}^n) \,, & \mathrm{res}(\mathcal{V}) \circ_n \mathrm{res}(\mathcal{W}) &= \mathrm{res}(\mathcal{V}_{<0}\mathfrak{L}^n\mathcal{W}_{<0}) \ . \end{aligned} \tag{5.9}$$

Let the u_n now depend on variables $t_1, t_2, \ldots$, and set $\partial = \partial_{t_1}$. The hierarchy (2.5) of ODEs,

$$f_{t_n} = f \circ_n f = -\mathrm{res}(\mathfrak{L}^n) \,, \qquad n = 1, 2, \ldots \ , \tag{5.10}$$

by use of the WNA structure implies

$$\begin{aligned} \partial_{t_n}(\mathrm{res}(\mathfrak{L}^m)) &= -\partial_{t_n}(f \circ_m f) = -f_{t_n} \circ_m f - f \circ_m f_{t_n} \\ &= -(f \circ_n f) \circ_m f - f \circ_m (f \circ_n f) = \mathrm{res}\Big(\mathfrak{L}^m (\mathfrak{L}^n)_{<0} - (\mathfrak{L}^n)_{\geq 0}\mathfrak{L}^m\Big) \\ &= \mathrm{res}\Big([(\mathfrak{L}^n)_{\geq 0}, \mathfrak{L}^m]\Big) \ . \end{aligned} \tag{5.11}$$

[14]If $m = 0$, the linear term is simply u_{n+2} and thus again 'the linear term with the highest derivative'.

[15]Note that the corresponding WNA structure for $(\mathcal{O}, \circ)$ resembles that of example 3 in section 3.

Since also $\partial_{t_n}(\mathrm{res}(\mathfrak{L}^m)) = \mathrm{res}([(\mathfrak{L}^m)_{\geq 0}, \mathfrak{L}^n]) = \partial_{t_m}(\mathrm{res}(\mathfrak{L}^n))$, we conclude that if we extend $\mathcal{A}$ to $\tilde{\mathcal{A}}$ by adjoining an element $\phi = \partial^{-1}(u_2)$, then

$$\phi_{t_n} = \mathrm{res}(\mathfrak{L}^n)\,, \qquad n = 1, 2, \ldots\ . \tag{5.12}$$

It follows that $\nu := f + \phi$ satisfies $\partial_{t_n}(\nu) = 0$, $n = 1, 2, \ldots$, and is therefore constant. (5.12) determines all the u_k in terms of the derivatives of ϕ (see [67], for example). From (5.12) with $n = 2, 3$, and (5.11) with $m = n = 2$, we recover the pKP equation

$$(4\phi_{t_3} - \phi_{t_1 t_1 t_1} - 6\,{\phi_{t_1}}^2)_{t_1} - 3\,\phi_{t_2 t_2} + 6\,[\phi_{t_1}, \phi_{t_2}] = 0\,, \tag{5.13}$$

in accordance with the general theory. More generally, the equations (5.11) determine the whole pKP hierarchy. They are the residues of

$$\partial_{t_n}(\mathfrak{L}^m) = [(\mathfrak{L}^n)_{\geq 0}, \mathfrak{L}^m]\,, \qquad m, n = 1, 2, \ldots\ . \tag{5.14}$$

This is equivalent to the Gelfand-Dickey (GD) system $\partial_{t_n}(\mathfrak{L}) = [(\mathfrak{L}^n)_{\geq 0}, \mathfrak{L}]$, $n = 1, 2, \ldots$, which is a well-known formulation of the KP hierarchy (see [5], for example).

We have thus shown how the Gelfand-Dickey formulation of the KP hierarchy can be recovered in the WNA framework. In fact, for the particular WNA algebra chosen above, the hierarchy (2.5) of ODEs is equivalent to the Gelfand-Dickey formulation of the KP hierarchy.

6 Conclusions

In this work we extended our previous results [15, 61] on the relation between weakly nonassociative (WNA) algebras and solutions of KP hierarchies to discrete KP hierarchies. We also provided further examples of solutions of matrix KP hierarchies and corresponding solutions of the scalar KP hierarchy. In particular we recovered a well-known tau function related to Calogero-Moser systems in this way (example 2 in section 3). Furthermore, we established a connection with the Gelfand-Dickey formulation of the KP hierarchy. As a byproduct, in section 5 we obtained a new realization of the *free* WNA algebra generated by a single element, which also has a realization in terms of quasi-symmetric functions [15]. There is more, however, we have to understand in the WNA framework. In particular this concerns the multi-component KP hierarchy (see [68] and references therein) and its reductions, which include the Davey-Stewartson, two-dimensional Toda lattice and N-wave hierarchies. Our hope is that also in these cases the WNA approach leads in a quick way to relevant classes of exact solutions.

Appendix: From Riccati to KP with FORM

The following FORM program [44,45] verifies that any solution of the first three equations of the Riccati hierarchy (3.5) solves the pKP equation in an algebra with product $A \circ B = AQB$.

```
Functions phi,phix,phiy,phit,L,Q,R,S,dx,dy,dt;  Symbol n;
Local pKP = dx*(4*phit - 6*phix*Q*phix - dx^2*phix) - 3*dy*phiy
 + 6*( phix*Q*phiy - phiy*Q*phix );       * pKP equation
repeat;
id phix = S + L*phi - phi*R - phi*Q*phi;  * Riccati system
id phiy = S(2) + L(2)*phi - phi*R(2) - phi*Q(2)*phi;
id phit = S(3) + L(3)*phi - phi*R(3) - phi*Q(3)*phi;
id dx*phi = phix + phi*dx;   * product rule of differentiation
```

```
id dy*phi = phiy + phi*dy;  id dt*phi = phit + phi*dt;
id dx?{dx,dy,dt}*L?{L,Q,R,S} = L*dx;      * L,Q,R,S are constant
* recursion relations for matrices (see proof of proposition 2):
id L(n?{2,3}) = L*L(n-1) + S*Q(n-1);
id R(n?{2,3}) = Q*S(n-1) + R*R(n-1);
id S(n?{2,3}) = L*S(n-1) + S*R(n-1);
id Q(n?{2,3}) = Q*L(n-1) + R*Q(n-1);
id L?{L,Q,R,S}(1) = L;
endrepeat;
id dx?{dx,dy,dt} = 0;
print pKP;          * should return zero
.end
```

This program provides an elementary and quick way toward the classes of exact solutions of the KP equation given in the examples in section 3.

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