Exact solvability of superintegrable Benenti systems

Artur Sergyeyev[*]

Silesian University in Opava, Mathematical Institute,

Na Rybníčku 1, 746 01 Opava, Czech Republic

July 31, 2018

We establish quantum and classical exact solvability for two large classes of maximally superintegrable Benenti systems in n dimensions with arbitrarily large n. Namely, we solve the Hamilton–Jacobi and Schrödinger equations for the systems in question. The results obtained are illustrated for a model with the cubic potential.

---

[*]Electronic mail: Artur.Sergyeyev@math.slu.cz

I   Introduction

Completely integrable systems in classical mechanics are well known to be of great interest for both theory and applications, see e.g. [1, 2] and references therein. Indeed, the possibility of analytical description of the corresponding dynamics enables us to uncover important physical properties of the systems in question. The prime examples of this are the Kepler laws and numerous physical models based on (superposition of) harmonic oscillators.

Interestingly, a number of physically relevant exactly solvable models (e.g., the Coulomb problem for the hydrogen atom and the multidimensional harmonic oscillator, to name just a few) are *maximally superintegrable* rather than just completely integrable. In general, a Hamiltonian dynamical system on a $2n$-dimensional phase space is *maximally superintegrable*, if it possesses the maximal possible number, $2n - 1$, of functionally independent, globally defined integrals of motion (contrast this with n *commuting* integrals of motion for completely integrable systems); see e.g. [3, 4] and references therein for further details. In this case we shall also say that the Hamiltonian of the system in question is *maximally superintegrable.*

Quantizing a generic completely integrable classical system and solving the resulting Schrödinger equation may often represent a nontrivial problem. However, solving the Schrödinger equation for maximally superintegrable systems is often easier (sometimes to the extent of reducing the determination of energy spectrum to a purely algebraic problem, as is e.g. the case for the nonrelativistic hydrogen atom and the multidimensional harmonic oscillator) because of the presence of additional integrals of motion [5].

These facts have lead to a considerable interest in superintegrable systems in general, and maximally superintegrable systems with natural Hamiltonians on two- or three-dimensional configuration space are now quite well understood, see e.g. [6]–[16] and the survey [4]. However, much less is known about superintegrable systems in higher dimensions, although such systems often can be interpreted as multiparticle systems consisting of several one-, two- or three-dimensional particles with nontrivial interactions and therefore also can be of interest in physics.

Moreover, quantum systems on an $2n$-dimensional phase space that are exactly solvable for arbitrary n are relatively scarce, with a few beautiful exceptions like the multidimensional Coulomb problem and its extensions [17] and the Calogero–Moser–Sutherland and related models, see e.g. [18] and references therein. Therefore, any new examples would be of considerable interest. Higher-dimensional superintegrable systems are prime candidates to yield such examples: for instance, the above examples are superintegrable, see e.g. [3, 17, 19] and references therein. In the present

work we show that for two large classes of maximally superintegrable Hamiltonians on an $2n$-dimensional phase space with arbitrary n the exact solutions for the Hamilton–Jacobi and the Schrödinger equation are readily available, see Theorems 3 and 4 below for details.

More specifically, we show that solving the Hamilton–Jacobi and the Schrödinger equations for the systems in question on the n-dimensional configuration space with arbitrary n amounts (provided n is sufficiently large) to solving k-dimensional *reduced* equations, where k is fixed and independent of n, as presented in Theorems 3 and 4. Moreover, the said reduced equations can be readily solved, as illustrated by the example of an anharmonic oscillator in Section V.

In order to establish these results, we reveal a hidden symmetry of the systems under study. This symmetry manifests itself upon passing to the flat coordinates of the metric tensors associated with the kinetic-energy parts of the Hamiltonians under study, as described in Theorem 2 below. Namely, some of these flat coordinates are cyclic coordinates for the Hamiltonians under study, so the corresponding momenta are integrals of motion, and the separation of variables is a reduced one in the sense of [21]. The dynamics in the non-cyclic coordinates is described by the reduced equations mentioned above.

Note that the metric tensors in question have the so-called maximally balanced signature (the numbers of plus and minus signs differ at most by one), and the above flat coordinates are the light-cone ones rather than the orthogonal ones. This leads to an interesting phenomenon: the aforementioned reduced equations are *first-order* rather than second-order PDEs, which makes them easier to solve.

II Preliminaries

For fixed integer m and k consider the separable Benenti Hamiltonians [22, 23] on the phase space $T^*\mathcal{Q}$, the cotangent space of an n-dimensional Riemannian manifold $\mathcal{Q}$ endowed with the contravariant metric tensor G_m:

$$H_r^{(m,k)} = \frac{1}{2}\mu^T K_r G_m \mu + V_r^{(k)} \quad r = 1, \ldots, n. \tag{1}$$

Here $\lambda = (\lambda^1, \ldots, \lambda^n)^T$ are coordinates on $\mathcal{Q}$ and $\mu = (\mu_1, \ldots, \mu_n)^T$ are the corresponding momenta. The contravariant metric tensors G_m have the form [24]

$$G_m = L^m G_0, \quad m \in \mathbb{Z}, \quad G_0 = \mathrm{diag}\left(\frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n}\right),$$

where $\Delta_i = \prod_{j \neq i}(\lambda^i - \lambda^j)$, and $L = \mathrm{diag}(\lambda^1, \ldots, \lambda^n)$ is a $(1,1)$-tensor on $\mathcal{Q}$ called a *special conformal Killing tensor* [25]. The Killing tensors K_r from (1) are diagonal in the λ-coordinates and can be

3

constructed as follows [22]:

$$K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} (-1)^k \sigma_k L^{r-1-k}, \quad r = 2, \ldots, n, \tag{2}$$

where $\mathbb{I}$ is an $n \times n$ unit matrix, and $\sigma_k = \sigma_k(\lambda)$ are symmetric polynomials in the variables $\lambda^1, \ldots, \lambda^n$ ($\sigma_0 = 1$, $\sigma_1 = \sum_{i=1}^{n} \lambda^i, \ldots, \sigma_n = \lambda^1 \lambda^2 \cdots \lambda^n$). They are related to the coefficients of the characteristic polynomial of the tensor L as follows:

$$\det(\xi \mathbb{I} - L) = \sum_{i=0}^{n} (-1)^i \sigma_i \xi^{n-i}. \tag{3}$$

Consider the (q, p)-coordinates defined as follows [20]:

$$q^i = (-1)^i \sigma_i(\lambda), \quad p_i = -\sum_{k=1}^{n} (\lambda^k)^{n-i} \mu_k / \Delta_k, \quad i = 1, \ldots, n. \tag{4}$$

Notice that q^i are nothing but the coefficients of the characteristic polynomial of L (3). In the (q, p)-coordinates we have

$$(G_0)^{rs} = \delta_{n+j}^{r+s} + \sum_{j=1}^{n-1} q^j \delta_{n+j}^{r+s}, \quad L_j^i = -\delta_j^1 q^i + \delta_j^{i+1},$$

whence for $m = 0, \ldots, n$ we find [26]

$$(G_m)^{rs} = \begin{cases} \delta_{n-m+1}^{r+s} + \sum_{j=1}^{n-m-1} q^j \delta_{n-m+j+1}^{r+s}, & r, s = 1, \ldots, n-m, \\[2mm] -\sum_{j=n-m+1}^{n} q^j \delta_{n-m+j+1}^{r+s}, & r, s = n-m+1, \ldots, n, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

Here and below δ_i^j stands for the Kronecker delta.

The geodesic Hamiltonians $E_{m,r} = \frac{1}{2} \mu^T K_r G_m \mu$ are polynomial in the (p, q)-coordinates for $m = 0, 1, 2, \ldots$ [20]. In particular, for $m = 0, \ldots, n$ we have

$$E_{m,1} = \frac{1}{2} \sum_{j=1}^{n-m} p_j p_{n-m-j+1} + \frac{1}{2} \sum_{k=1}^{n-m-1} q^k \sum_{j=k+1}^{n-m} p_j p_{n-m+k-j+1} - \frac{1}{2} \sum_{k=1}^{m} q^{n-m+k} \sum_{j=1}^{k} p_{n-m+j} p_{n-m+k-j+1}.$$

The basic separable potentials are given by the recursion relations [20]

$$V_r^{(k+1)} = V_{r+1}^{(k)} + V_r^{(1)} V_1^{(k)}, \quad k = 1, 2, \ldots, \quad V_r^{(1)} = -q^r,$$
$$V_r^{(0)} = 0, \tag{5}$$
$$V_r^{(-k-1)} = V_{r-1}^{(-k)} + V_r^{(-1)} V_n^{(-k)}, \quad k = 1, 2, \ldots, \quad V_r^{(-1)} = -q^{r-1}/q^n,$$

where we tacitly assume that $q^0 \equiv 1$ and that $q^i \equiv 0$ for $i > n$.

4

For any fixed m and k the Hamiltonians $H_r^{(m,k)}$ are in involution with respect to the canonical Poisson bracket on $T^*\mathcal{Q}$

$$\{f,g\} = \sum_{j=1}^{n} \frac{\partial f}{\partial \lambda^j}\frac{\partial g}{\partial \mu_j} - \frac{\partial g}{\partial \lambda^j}\frac{\partial f}{\partial \mu_j}.$$

These Hamiltonians are automatically separable in the (λ,μ)-coordinates because they satisfy the Stäckel separation relations by construction [24]. Note that the transition from the (λ,μ)- to the (p,q)-coordinates is a canonical transformation.

III Superintegrability and flat coordinates

For any natural number $n \geq 2$ let

$$H_{m,r} \equiv E_{m,r} + \sum_{k=k_0}^{k_1} c_k V_r^{(k)}, \quad r = 1,\ldots,n,$$

where c_k are arbitrary constants and m, k_0 and k_1 are integers. It is readily seen that these Hamiltonians are in involution for any fixed m, k_0 and k_1 (see Theorem 1 of [27] for details):

$$\{H_{m,r}, H_{m,s}\} = 0, \quad r,s = 1,\ldots,n.$$

We have the following straightforward generalization of Theorem 1 from [20] (see also [27]):

Theorem 1 *Given a natural $n \geq 2$ and an integer $m \in \{0,\ldots,n-1\}$, let $k_0 = -m$ and $k_1 = n - 1 - m$. Then p_{n-m} is an integral of motion for $H_{m,1}$, i.e., $\{H_{m,1}, p_{n-m}\} = 0$.*

Moreover, $F_{m,s} = \{H_{m,s}, p_{n-m}\}$ are additional integrals of motion for $H_{m,1}$: $\{H_{m,1}, F_{m,s}\} = 0$, $s = 2,\ldots,n$, and the $(2n-1)$ integrals of motion for $H_{m,1}$ ($H_{m,r}$, $r = 1,\ldots,n$, and $F_{m,s}$, $s = 2,\ldots,n$) are functionally independent, so $H_{m,1}$ is maximally superintegrable.

Under the assumptions of Theorem 1 the Hamiltonian $H_{m,1}$ involves $n-1$ parameters c_i (note that $V_1^{(0)} \equiv 0$).

Proposition 1 ([26]) *Given a natural $n \geq 2$ and an integer $m \in \{0,\ldots,n\}$, the metric G_m in the coordinates r^i defined by the formulas*

$$
\begin{aligned}
q^i &= r^i + \tfrac{1}{4}\sum_{j=1}^{i-1} r^j r^{i-j}, & i &= 1,\ldots,n-m, \\
q^i &= -\tfrac{1}{4}\sum_{j=i}^{n} r^j r^{n-j+i}, & i &= n-m+1,\ldots,n,
\end{aligned}
\tag{6}
$$

takes the form

$$(G_m)^{kl} = \left(\delta_{n-m+1}^{k+l} + \delta_{2n-m+1}^{k+l}\right).\tag{7}$$

The transition from (q,p)- to (r,π)-coordinates, where $\pi_k = \sum\limits_{i=1}^{n} \dfrac{\partial q^i}{\partial r^k} p_i$, $k = 1,\ldots,n$, is a canonical transformation, and we have

$$E_{m,1} \equiv H_{m,1}^{(0)} = \frac{1}{2}\left(\sum_{j=1}^{n-m} \pi_j \pi_{n-m+1-j} + \sum_{j=n-m+1}^{n} \pi_j \pi_{2n-m+1-j} \right). \tag{8}$$

The tensor L in the coordinates r^i takes the form:

$$\text{for } m < n: \quad L_j^i = \delta_j^{i+1}(1 - \delta_{n-m}^i) - \tfrac{1}{2}r^i\delta_j^1 - \tfrac{1}{2}r^{n-j-m+1+n[(j+m-1)/n]}\delta_{n-m}^i$$

$$\text{for } m = n: \quad L_j^i = \delta_j^{i+1} + \tfrac{1}{4}r^i r^{n-j+1}.$$

Here and below $[k]$ denotes the largest integer less than or equal to k.

The canonical coordinates (r,π) are not orthogonal, but the metric tensor G_m is constant in these coordinates. Bringing G_m into the canonical form, with $+1$ and -1 at the diagonal and zeros off the diagonal, is possible [26] but we shall not need this here.

Recall [20] that for $k = 1,\ldots,n-1$ the potentials $V_1^{(k)}$ are independent of q^j with $j = k+1,\ldots,n$. Likewise, for $k = 1,\ldots,n-1$ the potentials $V_1^{(-k)}$ are independent of q^j with $j = 1,\ldots,n-k$. On the other hand, the change of variables (6) is partially triangular: q^i with $i = 1,\ldots,n-m$ depend only on $r^1,\ldots,r^i$ while q^i with $i = n-m+1,\ldots,n$ depend only on $r^i,\ldots,r^n$.

Hence the coordinates r^i enjoy the following remarkable property:

Theorem 2 *Given a natural $n \geq 2$, and two non-negative integers, $m \in \{0,\ldots,n-2\}$ and k, consider the Hamiltonians $H_{m,1}^{(k,+)} = E_{m,1} + \sum\limits_{j=1}^{k} c_j V_1^{(j)}$ and $H_{m,1}^{(-k,-)} = E_{m,1} + \sum\limits_{j=1}^{k} c_j V_1^{(-j)}$, where c_j are arbitrary constants.*

If $k \in \{1,\ldots,n-m\}$ then the Hamiltonian $H_{m,1}^{(k,+)}$ commutes not only with the 'standard' integrals $H_{m,r}^{(k,+)} = E_{m,r} + \sum\limits_{j=1}^{k} c_j V_r^{(j)}$, $r = 2,\ldots,n$, but also with π_j, $j = k+1,\ldots,n$, i.e., r^j are cyclic variables for $H_{m,1}^{(k,+)}$ for $j = k+1,\ldots,n$.

Likewise, if $k \in \{0,\ldots,m\}$ then the Hamiltonian $H_{m,1}^{(-k,-)}$ commutes, in addition to $H_{m,r}^{(-k,-)} = E_{m,r} + \sum\limits_{j=1}^{k} c_j V_r^{(-j)}$, $r = 2,\ldots,n$, with π_j, $j = 1,\ldots,n-k$, i.e., r^j are cyclic variables for $H_{m,1}^{(-k,-)}$ for $j = 1,\ldots,n-k$.

Note that, in contrast with the above, the additional integrals of motion for $H_{i,1}^{(k)}$ found earlier in [20] were quadratic rather than linear in momenta.

IV Exact solvability in n dimensions

Because of the special form (7) of the metric G_m in the variables $r^1, \ldots, r^n$ the existence of cyclic variables simplifies solving the Hamilton–Jacobi and the Schrödinger equations for the Hamiltonians $H_{m,1}^{(k,+)}$ with $m = 0, \ldots, n-2$ and $k = 1, \ldots, n-m-1$ and $H_{m,1}^{(-k,-)}$ for $m = 0, \ldots, n-2$ and $k = 0, \ldots, m$ even more than one could expect, *especially* provided n is sufficiently large.

Let $\mathcal{P}_s \equiv -\mathrm{i}\partial/\partial r^s$, $s = 1, \ldots, n$, where $\mathrm{i} = \sqrt{-1}$. The following results are readily verified by straightforward computation:

Theorem 3 *Fix a non-negative integer m, a natural k, and k constants c_j, $j = 1, \ldots, k$. Then for any natural n such that $n \geq 2k + m$ the most general common eigenfunction ψ of*

$$\mathcal{H}_{m,1}^{(k,+)} \equiv \frac{1}{2}\left(\sum_{a=1}^{n-m} \mathcal{P}_a \mathcal{P}_{n-m+1-a} + \sum_{b=n-m+1}^{n} \mathcal{P}_b \mathcal{P}_{2n-m+1-b}\right) + \sum_{j=1}^{k} c_j V_1^{(j)}$$

and of $\mathcal{P}_j$, $j = k+1, \ldots, n$, with the eigenvalues E and π_j, $j = k+1, \ldots, n$, respectively, is quasiclassical and reads $\psi = \exp(\mathrm{i}S)$, where $S(r^1, \ldots, r^n) = S_0(r^1, \ldots, r^k) + \sum_{j=k+1}^{n} r^j \pi_j$ satisfies the stationary Hamilton–Jacobi equation for $H_{m,1}^{(k,+)}$, and S_0 is a general solution of the reduced Hamilton–Jacobi equation, a first order linear PDE in k independent variables:

$$\sum_{j=1}^{k} \pi_{n-m+1-j} \frac{\partial S_0}{\partial r^j} + \sum_{j=1}^{k} c_j V_1^{(j)} = \varepsilon, \tag{9}$$

where

$$\varepsilon = E - \sum_{j=k+1}^{[(n-m)/2]} \pi_{n-m+1-j}\pi_j - \frac{(n-m-2[(n-m)/2])}{2}\pi_{n-m-[(n-m)/2]}^2 - \frac{1}{2}\sum_{j=n-m+1}^{n} \pi_j \pi_{2n-m+1-j}.$$

Theorem 4 *Fix a natural k, a non-negative integer $m \geq 2k$, and k constants c_j, $j = 1, \ldots, k$. Then for any natural $n \geq m$ the most general common eigenfunction ψ of*

$$\mathcal{H}_{m,1}^{(-k,-)} \equiv \frac{1}{2}\left(\sum_{a=1}^{n-m} \mathcal{P}_a \mathcal{P}_{n-m+1-a} + \sum_{b=n-m+1}^{n} \mathcal{P}_b \mathcal{P}_{2n-m+1-b}\right) + \sum_{j=1}^{k} c_j V_1^{(-j)}$$

and of $\mathcal{P}_j$, $j = 1, \ldots, n-k$, with the eigenvalues E and π_j, $j = 1, \ldots, n-k$, respectively, is quasiclassical and reads $\psi = \exp(\mathrm{i}S)$, where $S(r^1, \ldots, r^n) = S_0(r^{n-k+1}, \ldots, r^n) + \sum_{j=1}^{n-k} r^j \pi_j$ satisfies the stationary Hamilton–Jacobi equation for $H_{m,1}^{(-k,-)}$, and S_0 is a general solution of the reduced Hamilton–Jacobi equation, a first order linear PDE in k independent variables:

$$\sum_{j=n-k+1}^{n} \pi_{2n-m+1-j} \frac{\partial S_0}{\partial r^j} + \sum_{j=1}^{k} c_j V_1^{(-j)} = \tilde{\varepsilon}, \tag{10}$$

where

$$\tilde{\varepsilon} = E - \frac{1}{2}\sum_{j=1}^{n-m} \pi_{n-m+1-j}\pi_j - \sum_{j=n-[m/2]+1}^{n-k} \pi_j \pi_{2n-m+1-j} - \frac{(m-2[m/2])}{2}\pi_{n-[m/2]}^2.$$

7

Let us briefly outline the integration strategy for (9) and (10). Consider (9) first. Assume that we have found new coordinates $z^1(r^1, \ldots, r^k), \ldots, z^k(r^1, \ldots, r^k)$ such that

$$\sum_{j=1}^{k} \pi_{n-m+1-j} \frac{\partial}{\partial r^j} = \frac{\partial}{\partial z^k}.$$

This is always possible, but the choice of z's is not unique and depends on the particular values of π_j. Now (9) becomes an *ODE* in z^k involving $z^1, \ldots, z^{k-1}$ as parameters:

$$\frac{\partial S_0}{\partial z^k} + \sum_{j=1}^{k} c_j V_1^{(j)} = \varepsilon, \tag{11}$$

where $V_1^{(j)}$ are now considered as functions of z's.

The general solution of (11) reads

$$S_0 = K(z^1, \ldots, z^{k-1}) - \sum_{j=1}^{k} \int c_j V_1^{(j)} dz^k + \varepsilon z^k, \tag{12}$$

where K is an arbitrary smooth function of its arguments.

Likewise, assume that we have found new coordinates $\tilde{z}^{n-k+1}(r^{n-k+1}, \ldots, r^n), \ldots, \tilde{z}^n(r^{n-k+1}, \ldots, r^n)$ such that

$$\sum_{j=n-k+1}^{n} \pi_{2n-m+1-j} \frac{\partial}{\partial r^j} = \frac{\partial}{\partial \tilde{z}^n}.$$

Then the general solution of (10) reads

$$S_0 = \tilde{K}(\tilde{z}^{n-k+1}, \ldots, \tilde{z}^{n-1}) - \sum_{j=1}^{k} \int c_j V_1^{(-j)} d\tilde{z}^n + \tilde{\varepsilon} \tilde{z}^n, \tag{13}$$

where $\tilde{K}$ is an arbitrary smooth function of its arguments.

Thus, the stationary Schrödinger equations for the Hamiltonians $H_{m,1}^{(k,+)}$ and $H_{m,1}^{(-k,-)}$ with $m = 0, \ldots, n-2$ (for $H_{m,1}^{(-k,-)}$ we have an extra condition $m \geq 2k$) and arbitrary constants c_i, $i = 1, \ldots, k$, in the space of n dimensions for *any* $n \geq 2k + m$ essentially reduce to a linear *first-order* PDEs in k independent variables, and these PDE can be explicitly solved. This is a rather surprising result, as only a few potentials for which the Schrödinger equation is exactly solvable in the space of arbitrarily high dimension were known so far, cf. the discussion in Introduction.

It is readily seen that there exists no real value of E for which the eigenfunction

$$\psi = \exp\left(\mathrm{i}\left(S_0 + \sum_{j=k+1}^{n} \pi_j r^j\right)\right)$$

of $\mathcal{H}_{m,1}^{(k,+)}$ constructed in Theorem 3 with S_0 given by (12) can have a finite norm

$$\int_{\mathcal{Q}} |\psi|^2 dr^1 \ldots dr^n \sim \int_{\mathcal{Q}} |\exp(K(z^1, \ldots, z^{k-1}))|^2 dz^1 \ldots dz^n.$$

The latter integral obviously diverges for any choice of K because of integration over $z^{k+1}, \ldots,$ z^n. Therefore no common eigenfunction ψ of $\mathcal{H}_{m,1}^{(k,+)}$ and $\mathcal{P}_j$, $j = k+1, \ldots, n$, can belong to the discrete spectrum of $\mathcal{H}_{m,1}^{(+)}$. In a similar fashion we can show that no common eigenfunction ψ of $\mathcal{H}_{m,1}^{(-k,-)}$ and $\mathcal{P}_j$, $j = 1, \ldots, n-k$, can belong to the discrete spectrum of $\mathcal{H}_{m,1}^{(-k,-)}$.

V   Example: an anharmonic oscillator

Consider the Hamiltonian

$$H \equiv H_{0,1}^{(3)} = \frac{1}{2} \sum_{k=0}^{n-1} q^k \sum_{j=k+1}^{n} p_j p_{n+k-j+1} - q^3 + 2q^1 q^2 - (q^1)^3,$$

in the (q, p)-cootdinates. It commutes with

$$H_i \equiv H_{0,i}^{(3)} = \frac{1}{2} \sum_{i,j=1}^{n} (K_r G)^{ij} p_i p_j - q^{i+2} + q^{i+1} q^1 + q^i q^2 - q^i (q^1)^2, \quad i = 2, \ldots, n,$$

by construction.

A   Superintegrability

The Hamiltonian H is superintegrable [20] because it also commutes with the function

$$I = \begin{cases} \frac{1}{2} p_2^2 + (q^1)^2, & n = 2, \\ \frac{1}{2} p_3^2 - q^1, & n = 3, \\ p_n, & n \geq 4, \end{cases}$$

and hence $F_r = \{I, H_r\}$, $r = 2, \ldots, n$ also Poisson commute with H, and all functions in the set $\{H_r, r = 1, \ldots, n, F_s, s = 2, \ldots, n\}$ are functionally independent.

For $n \geq 4$ the additional integrals F_r are simply $F_r = \{p_n, H_r\} = -\partial H_r / \partial q^n$, and we readily find that

$$F_r = \frac{1}{2} \sum_{i=n-r+2}^{n} \sum_{j=2n+2-i-r}^{n} q^{i+j+r-2n-2} p_i p_j + \delta_n^{r+2} - \delta_n^{r+1} q^1 - \delta_n^r (q^2 - (q^1)^2).$$

In particular, we obtain

$$F_2 = p_n^2 / 2, \tag{14}$$

so F_2 is simply a half of square of I, and it is straightforward to verify (see Theorem 5 of [27] for a more general result of this kind) that we have

$$\{p_n, F_r\} = 0, \quad r = 2, \ldots, n, \quad \{F_r, F_s\} = 0, \quad r, s = 2, \ldots, n.$$

9

It can be readily inferred from the above and from Theorem 1 that the quantities $\{H, I, F_3, \ldots, F_n\}$ are functionally independent and Poisson commute for all $n \geq 2$.

Thus, the Hamiltonian H is maximally superintegrable for all $n = 2, 3, \ldots$. For $n = 3$ we have, in addition to H, H_2, H_3 and I, the following integral $K = p_3 p_2 + q^1 p_3^2/2 - q^2 + (q^1)^2 = \pi_2 \pi_3 - r^2 + 3(r^1)^2/4$ which is quadratic in momenta. Note that I and K commute.

B Quantization and solution of equations of motion in the (r, π)-coordinates

In the (r, π)-coordinates we have

$$H = \frac{1}{2} \sum_{j=1}^{n} \pi_j \pi_{n+1-j} - r^3 + \frac{3}{2} r^1 r^2 - \frac{1}{2}(r^1)^3.$$

The quantization is obvious: H goes into the operator

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^{n} \mathcal{P}_j \mathcal{P}_{n+1-j} - r^3 + \frac{3}{2} r^1 r^2 - \frac{1}{2}(r^1)^3.$$

For $n \geq 4$ we can look for the common eigenfunctions ψ of $\mathcal{H}$ and $\mathcal{P}_i$, $i = 4, \ldots, n$:

$$\mathcal{H}\psi = E\psi, \quad \mathcal{P}_i \psi = \pi_i \psi, \quad i = 4, \ldots, n. \tag{15}$$

By Proposition 2, for $n \geq 6$ finding ψ requires solving the equation (9) that becomes

$$\sum_{k=1}^{3} \pi_{n+1-k} \partial S_0 / \partial r^k - r^3 + \tfrac{3}{2} r^1 r^2 - \tfrac{1}{2}(r^1)^3 = \varepsilon, \tag{16}$$

where

$$\varepsilon = E - \sum_{k=4}^{[n/2]} \pi_{n+1-k} \pi_k - \frac{(n - 2[n/2])}{2} \pi_{[n/2]+1}^2.$$

Eq.(16) is a *first* order PDE that can be readily solved in full generality.

Let $K(\omega_1, \omega_2)$ stand below for an arbitrary (smooth) function of its arguments.

If $\pi_n \neq 0$ then the general solution of (16) reads

$$S_0 = \frac{(r^1)^4}{8\pi_n} + \frac{\pi_{n-1}(r^1)^3}{4\pi_n^2} + \left(-\frac{3r^2}{4\pi_n} - \frac{\pi_{n-2}}{2\pi_n^2} \right)(r^1)^2 + \frac{(r^3 + \varepsilon)r^1}{\pi_n}$$

$$+ K\left(\frac{(r^2 \pi_n - \pi_{n-1} r^1)}{\pi_n}, \frac{(r^3 \pi_n - \pi_{n-2} r^1)}{\pi_n} \right).$$

If $\pi_n = 0$ but $\pi_{n-1} \neq 0$ then the general solution of (16) is

$$S_0 = \frac{(r^1)^3 r^2}{2\pi_{n-1}} - \frac{3r^1 (r^2)^2}{4\pi_{n-1}} - \frac{\pi_{n-2}(r^2)^2}{2\pi_{n-1}^2} + \frac{(r^3 + \varepsilon)r^2}{\pi_{n-1}} + K\left(r^1, \frac{(r^3 \pi_{n-1} - \pi_{n-2} r^2)}{\pi_{n-1}} \right).$$

10

Finally, if $\pi_n = \pi_{n-1} = 0$ but $\pi_{n-2} \neq 0$ then the general solution of (16) has the form

$$S_0 = \frac{(r^3)^2 - 3r^1r^2r^3 + (r^1)^3r^3 + 2\varepsilon r^3}{2\pi_{n-2}} + K(r^1, r^2).$$

By Proposition 2 for $n \geq 6$ the most general common eigenfunction ψ of $\mathcal{H}$ and of $\mathcal{P}_i$, $i = 4, \ldots, n$ is

$$\psi = \exp\left(\mathrm{i}\left(S_0 + \sum_{j=4}^{n} \pi_j r^j\right)\right)$$

where S_0 is given above. This eigenfunction is *not* square integrable for any choice of K and for any values of π_i.

An integral of the stationary Hamilton–Jacobi equation for H is given by

$$S = S_0 + \sum_{j=4}^{n} \pi_j r^j.$$

C Special cases: $n \leq 5$

We list below the exact solutions for the corresponding Schrödinger and Hamilton–Jacobi equations. None of the eigenfunctions listed below is square integrable.

Case 1: $n = 2$

Here $q^3 = 0$ and there is a pair of commuting operators $\mathcal{H}$ and $\mathcal{I}$. In the (r, π)-coordinates we have

$$\mathcal{H} = \mathcal{P}_1\mathcal{P}_2 + 2r^1r^2 - (r^1)^3/2, \quad \mathcal{I} = \mathcal{P}_2^2/2 + (r^1)^2.$$

and their common eigenfunction ψ satisfying $\mathcal{H}\psi = E\psi$, $\mathcal{I}\psi = \lambda_1\psi$ has, up to multiplication by an arbitrary constant, the form

$$\psi = \frac{1}{\sqrt{(r^1)^2 - \lambda_1}}\left(r^1 + \sqrt{(r^1)^2 - \lambda_1}\right)^{E/\sqrt{2}} \exp\left(\sqrt{2((r^1)^2 - \lambda_1)}(r^2 - (r^1)^2/12 - \lambda_1/6)\right)$$

We actually have two such eigenfunctions, as $\sqrt{(r^1)^2 - \lambda_1}$ is a two-valued function.

The eigenfunction ψ is not quasiclassical: we have a complete integral of the stationary Hamilton–Jacobi equation for H of the form

$$S = -\mathrm{i}\ln\psi + \frac{\mathrm{i}}{3\sqrt{2}}((r^1)^2 - \lambda_1)^{1/2}(-2\lambda_1 - (r^1)^2 + 12r^2) - \frac{\mathrm{i}}{2}\ln((r^1)^2 - \lambda_1),$$

or equivalently,

$$S = \frac{E}{\sqrt{2}}\arctan\left(\frac{r^1}{((r^1)^2 - \lambda_1)^{1/2}}\right) - \frac{1}{12}(12r^2 - (r^1)^2 - 2\lambda_1)(2(\lambda_1 - (r^1)^2))^{1/2}.$$

11

Case 2: $n = 3$

Now we have a triplet of commuting operators $\mathcal{H}$, $\mathcal{I} = \pi_3^2/2 - r^1$, and $\mathcal{F}_2 = \mathcal{P}_3^3/2 - 3r^1\mathcal{P}_3/2 - \mathcal{P}_2$ and their common eigenfunction ψ such that $\mathcal{H}\psi = E\psi$, $\mathcal{I}\psi = \lambda_1\psi$ and $\mathcal{F}_2\psi = \lambda_2\psi$ reads

$$\psi = \frac{1}{(r^1 + \lambda_1)^{1/2}}\exp\Big(i\sqrt{2}(\lambda_1 + r^1)^{1/2}r^3 + i(-\lambda_2 + \sqrt{2}(r^1 + \lambda_1)^{1/2}\lambda_1)r^2$$

$$-\frac{i}{\sqrt{2}}(r^1 + \lambda_1)^{1/2}r^1r^2 + \frac{i}{28}\sqrt{2}(r^1 + \lambda_1)^{1/2}(r^1)^3 + \frac{i}{4}(-\lambda_2 + \frac{3}{7}\sqrt{2}(r^1 + \lambda_1)^{1/2}\lambda_1)(r^1)^2$$

$$-i(-\lambda_2 + \frac{\sqrt{2}}{7}(r^1 + \lambda_1)^{1/2}\lambda_1)\lambda_1 r^1 + \frac{i\sqrt{2}}{14}(r^1 + \lambda_1)^{1/2}(-10\lambda_1^3 + 14E - 7\lambda_2^2)\Big)$$

If we take another triple of commuting operators $\mathcal{H}$, $\mathcal{I} = \pi_3^2/2 - r^1$, and $\mathcal{K} = \mathcal{P}_2\mathcal{P}_3 - r^2 + 3(r^1)^2/4$ then their common eigenfunction ψ such that $\mathcal{H}\psi = E\psi$, $\mathcal{I}\psi = \lambda_1\psi$ and $\mathcal{K}\psi = \nu\psi$ reads

$$\psi = \frac{1}{(r^1 + \lambda_1)^{3/4}}\exp\Big(i\sqrt{2}(\lambda_1 + r^1)^{1/2}r^3 - \frac{i}{\sqrt{2}(\lambda_1 + r^1)^{1/2}}\Big(-\frac{23}{224}(r^1)^4$$

$$-\frac{\lambda_1}{28}(r^1)^3 + \frac{3}{4}r^2(r^1)^2 + \frac{1}{28}(-7\nu + 2\lambda_1^2)(r^1)^2 - \frac{1}{7}(14E + 2\lambda_1^3 - 7\lambda_1\nu)r^1 - (r^2)^2/2 - \nu r^2$$

$$-(-2\lambda_1^2\nu + \tfrac{4}{7}\lambda_1^4 + 2\lambda_1 E + \tfrac{1}{2}\nu^2)\Big)\Big).$$

In the latter case the eigenfunction ψ is almost quasiclassical: we have a complete integral of the Hamilton–Jacobi equation for H of the form

$$S = -i\ln\psi + \frac{3i}{4}\ln(r^1 + \lambda^1).$$

In both cases we again actually have two eigenfunctions as $(r^1 + \lambda_1)^{1/2}$ is two-valued.

Case 3: $n = 4$

The common eigenfunction of $\mathcal{P}_4$, $\mathcal{H}$ and $\mathcal{F}_i$, $i = 2,3$, with the respective eigenvalues π_4, E, λ_2,λ_3 reads

$$\psi = \exp\Big(i\pi_4 r^4 + \frac{i}{16\pi_4^5}(2\pi_4^4 + 2 + 3\pi_4^2)(r^1)^4 + \frac{i}{4\pi_4^5}\lambda_2(\pi_4^2 + 2)(r^1)^3 - \frac{i}{4\pi_4^3}(3\pi_4^2 + 2)(r^1)^2r^2$$

$$-\frac{i}{4\pi_4^5}(-3\lambda_2^2 + 2\pi_4^2\lambda_3)(r^1)^2 - \frac{i\lambda_2}{\pi_4^3}r^2r^1 + \frac{i}{\pi_4}r^3r^1 + \frac{i}{2\pi_4^5}(-2\lambda_2\pi_4^2\lambda_3 + \lambda_2^3 + 2\pi_4^4 E)r^1$$

$$+\frac{i}{2\pi_4}(r^2)^2 + \frac{i}{2\pi_4^3}(-\lambda_2^2 + 2\pi_4^2\lambda_3)r^2 + \frac{ir^3}{\pi_4}\lambda_2\Big)$$

This eigenfunction is quasiclassical: $S = -i\ln\psi$ is a complete integral for the stationary Hamilton–Jacobi equation for H.

Case 4: $n = 5$

The common eigenfunction of $\mathcal{P}_i$, $i = 4, 5$, $\mathcal{H}$ and $\mathcal{F}_i$, $i = 3, 4$, with the respective eigenvalues π_4, π_5, E, λ_3, λ_4 reads, up to an overall constant factor,

$$\psi = \exp\Big(\mathrm{i}\pi_5 r^5 + \mathrm{i}\pi_4 r^4 + \frac{\mathrm{i}}{8\pi_5}(r^1)^4 + \frac{\mathrm{i}}{12\pi_5^3}(3\pi_5\pi_4 + 2)(r^1)^3 - \frac{3\mathrm{i}}{4\pi_5}(r^1)^2 r^2$$

$$+\frac{\mathrm{i}}{4\pi_5^3}(2\lambda_3 + \pi_4^2)(r^1)^2 - \frac{\mathrm{i}}{\pi_5^2}\pi_4 r^2 r^1 + \frac{\mathrm{i}}{\pi_5}r^3 r^1$$

$$-\frac{\mathrm{i}}{8\pi_5^3}(-8E\pi_5^2 - 4\lambda_3^2 + 3\pi_4^4 + 8\pi_4\lambda_4\pi_5 - 4\pi_4^2\lambda_3)r^1 + \frac{\mathrm{i}}{2\pi_5}(r^2)^2$$

$$+\frac{\mathrm{i}}{2\pi_5^2}(2\lambda_4\pi_5 - 2\pi_4\lambda_3 + \pi_4^3)r^2 - \frac{\mathrm{i}}{2\pi_5}(\pi_4^2 - 2\lambda_3)r^3\Big)$$

Unlike the previous case, this wave function is not quasiclassical: there is a complete integral of the stationary Hamilton–Jacobi equation for H of the form

$$S = -\mathrm{i}\ln\psi + \frac{1}{24\pi_5^3}(\pi_4^2 - 2r^1 - 2\lambda_3)^3$$

Acknowledgements

References

[1] V.I. Arnold, *Mathematical methods of classical mechanics* (Springer, N.Y., 1997).

[2] M. Błaszak, *Multi-Hamiltonian Theory of Dynamical Systems* (Springer, Heidelberg, 1998).

[3] S. Wojciechowski, Phys. Lett. A **95**, 279 (1983).

[4] P. Winternitz, Superintegrable systems in classical and quantum mechanics, in *New trends in integrability and partial solvability* (Kluwer, Dordrecht, 2004), pp.281–297.

[5] P. Tempesta, A.V. Turbiner, P. Winternitz, J. Math. Phys. **42**, 4248 (2001) [preprint hep-th/0011209].

[6] J. Friš, V. Mandrosov, Ya.A. Smorodinsky, M. Uhlíř and P. Winternitz, Phys. Lett. **16**, 354 (1965).

[7] N.W. Evans, Phys. Rev. A **41**, 5666 (1990).

[8] E.G. Kalnins, C.G. Williams, W. Miller, Jr and G.S. Pogosyan, J. Math. Phys. **40**, 708 (1999).

[9] M.F. Rañada and M. Santander, J. Math. Phys., **40**, 5026 (1999).

[10] E.G. Kalnins, W. Miller, Jr and G.S. Pogosyan, J. Phys. A: Math. Gen. **33**, 6791 (2000).

[11] E.G. Kalnins, J.M. Kress, G.S. Pogosyan and W. Miller, Jr, J. Phys. A: Math. Gen. **34**, 4705 (2001) [preprint math-ph/0102006].

[12] S. Gravel and P. Winternitz, J. Math. Phys. **43**, 5902 (2002) [preprint math-ph/0206046].

[13] E.G. Kalnins, J.M. Kress and P. Winternitz, J. Math. Phys. **43**, 970 (2002) [preprint math-ph/0108015]

[14] E.G. Kalnins, J.M. Kress, W. Miller, Jr, J. Math. Phys. **46**, 053509 (2005).

[15] A. Ballesteros, F.J. Herranz, M. Santander and T. Sanz-Gil, J. Phys. A: Math. Gen. **36**, L93 (2003) [preprint math-ph/0211012].

[16] R.G. Smirnov and P. Winternitz, J. Math. Phys. **47**, 093505 (2006) [preprint math-ph/0606006].

[17] M.A. Rodriguez and P. Winternitz, J. Math. Phys., **43**, 1309 (2002) [preprint math-ph/0110018].

[18] A.M. Perelomov, *Integrable systems of classical mechanics and Lie algebras* (Birkhäuser, Basel, 1990).

[19] C. Gonera, J. Phys. A: Math. Gen. **31**, 4465 (1998).

[20] M. Błaszak and A. Sergyeyev, J. Phys. A: Math. Gen. **38**, L1 (2005) [preprint nlin.SI/0412018].

[21] S. Benenti, C. Chanu, G. Rastelli, J. Math. Phys. **43**, 5811 (2002).

[22] S. Benenti, Orthogonal separable dynamical systems, in *Differential geometry and its applications* (Opava, 1992) vol.1, (Silesian University in Opava, Opava, 1993), pp. 163–184; available online at `http://www.emis.de/proceedings/5ICDGA/`

[23] S. Benenti, J. Math. Phys. **38**, 6578 (1997).

[24] M. Błaszak, J. Phys. A: Math. Gen. **38**, 1667 (2005) [preprint nlin.SI/0312025].

[25] M. Crampin and W. Sarlet, J. Math. Phys. **42**, 4313 (2001).

[26] M. Błaszak and A. Sergyeyev, Phys. Lett. A **365**, 28 (2007) [preprint nlin.SI/0604022].

[27] A. Sergyeyev, More on superintegrability of the Benenti systems, in preparation.