INVERSE AC JOSEPHSON EFFECT FOR A FLUXON IN A LONG MODULATED JUNCTION

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ABSTRACT

We analyze motion of a fluxon in a weakly damped ac-driven long Josephson junction with a periodically modulated maximum Josephson current density. We demonstrate both analytically and numerically that a pure *ac* bias current can drive the fluxon at a resonant mean velocity determined by the driving frequency and the spatial period of the modulation, provided that the drive amplitude exceeds a certain threshold value. In the range of strongly "relativistic" mean velocities, the agreement between results of a numerical solution of the effective (ODE) fluxon equation of motion and analytical results obtained by means of the harmonic-balance analysis is fairly good; morever, a preliminary PDE result tends to confirm the validity of the collective-coordinate (PDE-ODE) reduction. At nonrelativistic mean velocities, the basin of attraction, in position-velocity space, for phase-locked solutions becomes progressively smaller as the mean velocity is decreased.

1 INTRODUCTION

The 'inverse ac Josephson effect' is the generation of dc voltage across the Josephson junction by ac bias current flowing through it. This effect was studied in detail for a point-like junction [\[1, 2\]](#page-12-0), as well as for shuttle motion of a single fluxon (Josephson vortex) in a long but finite junction with reflecting edges[[3\]](#page-12-0). In the present work, we consider a different dynamical regime which may also be interpreted as the inverse ac Josephson effect: steady progressive motion of a fluxon in the ac-driven weakly damped long junction with periodically modulated local parameters. In an experiment, this can be realized by periodically changing the junction's barrier thickness[[4\]](#page-12-0); in the simplest theoretical model it is usually assumed that the only local parameter which is modulated is the maximum Josephson current, while all others are kept constant[[5\]](#page-12-0). Thus, the ac-driven periodically modulated junction is described by the normalized, modified sine-Gordon equation in the standard notation:

$$
\phi_{tt} - \phi_{xx} + \sin \phi = \epsilon \sin(kx) \sin \phi - \alpha \phi_t + \gamma \sin(\omega t). \tag{1}
$$

where α is the dissipative constant, ϵ and k are the amplitude and wave number of the modulation, and γ and ω are the amplitude and frequency of the ac bias current driving the junction. We assume that the ac bias current is distributed uniformly along the junction, which is a reasonably realistic assumption for overlap- or annular-geometry junctions [\[6\]](#page-12-0).

In this work, our objective is to analyze the motion of a fluxon in the model of Eq. (1) . In the absence of the perturbing factors ($\epsilon = \alpha = \gamma = 0$), the fluxon is described by the well-known solution of the sine-Gordon equation:

$$
\phi(x,t) = 4 \tan^{-1} \left[\exp\left(-\frac{x - \xi(t)}{\sqrt{1 - V^2}} \right) \right],\tag{2}
$$

where V is the fluxon's velocity, and $\xi = Vt$ is the coordinate of its center. Following the perturbative approach which has proved to be very useful in the theory of long Josephson junctions [\[7](#page-12-0)], we will treat all the terms on the right-hand side of Eq. (1) as small perturbations. Then, using energy balance or, equivalently (in this case), momentum balance, it is straightforward to derive, at the lowest order of the perturbation theory, the following equation of motion for the fluxon:

$$
\frac{d}{dt}\left(\frac{\dot{\xi}}{\sqrt{1-\dot{\xi}^2}}\right) = -\frac{\epsilon k}{2}\sqrt{1-\dot{\xi}^2}\cos(k\xi) - \frac{\alpha\dot{\xi}}{\sqrt{1-\dot{\xi}^2}} + \frac{\pi\gamma}{4}\sin(\omega t)\,,\tag{3}
$$

where, in order to simplify the equation, it is assumed that

$$
\sqrt{1 - \dot{\xi}^2} \ll \frac{2\pi}{k},\qquad(4)
$$

i.e., a characteristic width of the fluxon is small in comparison with the modulation wavelength (in the opposite limit the coefficient in front of the first term on the righthand side of Eq. (3) would be exponentially small).

Eq. (3) is an effective equation of motion for a relativistic particle in the presence of potential, friction, and driving forces, corresponding, respectively, to the three terms on the right-hand side. Since the driving force has no dc (constant) part, it may seem that the particle will only oscillate with zero mean velocity. However, in this work we will demonstrate that, in the presence of the spatially periodic potential, the ac drive may support mean progressive motion of the particle, compensating the dissipation-induced braking. Earlier, it has been predicted analytically for models of the Toda-lattice (TL) [\[8\]](#page-12-0) and of the Frenkel-Kontorova (FK) [\[9](#page-12-0)] type, and shown numerically for the TL model in the so-called dual formulation[[10\]](#page-12-0) that ac drive may support progressive motion of a soliton in weakly damped nonlinear dynamical lattices; a crucial circumstance which made the effect possible was the spatial periodicity of the lattice. Moreover, the stable propagation of ac-driven solitons in a model electric transmission line described by the damped dual TL equation was detected experimentally in the work[[11](#page-12-0)]. Note that for the TL in its usual form the effect does not take place because the ac drive can provide only

for the balance of energy, but not for that of the momentum, while in the dual formulation the soliton's momentum is identically zero, and energy balance becomes sufficient [\[12](#page-12-0)]. In the FK model the balance of energy alone is, in principle, sufficient since the momentum is not conserved in this system even in the absence of dissipation and drive, but the threshold predicted by the theory[[9\]](#page-12-0) for the effect proves to be so high that it would be hard to observe in a numerical experiment [\[12\]](#page-12-0).

In the present work, we are dealing with the continuum model of Eq. (1). However, in the presence of a periodic potential, the ac drive allows the soliton in this damped continuum system to move at a nonzero mean velocity, the underlying mechanism being essentially the same as in the discrete lattices. Indeed, if the soliton is moving with a mean velocity \overline{V} through the periodic potential, this gives rise to the temporal frequency $k\overline{V}$ (recall that k is the wave number which determines the spatial periodicity). The condition

$$
k\overline{V} = \omega \,,\tag{5}
$$

where ω is the driving frequency, may give rise to a resonant transfer of energy from the ac driving field to the soliton. In turn, this supply of energy gives the soliton a chance to compensate the dissipative losses. In the case of the long Josephson junction, the mean dc voltage across the junction is proportional to the mean velocity of the fluxon. Thus, supporting the fluxon's progressive motion by the ac bias current may be regarded as another manifestation of the inverse ac Josephson effect.

In section 2 of this work, we will develop a fully analytical approach to investigate the ac-driven motion of the fluxon in the "ultrarelativistic" case, i.e., when the mean velocity is close enough to the junction's limit (Swihart) velocity (which is equal to one in the notation adopted). In section 3, we will display results of the full numerical solution of the ODE, Eq. (3).

2 ANALYTICAL CONSIDERATIONS

An analytical approach to Eq. (3) is possible if one assumes that the small parameters α and γ are "very small", while ϵ is "not so small" (the physically relevant range for ϵ is $0 \leq \epsilon \leq 1$). In other words, we will consider, in the lowest approximation, motion of the free relativistic particle in the periodic potential, and then the friction and driving forces will be taken into account as additional small perturbations. Another fundamental assumption which makes the analytical consideration possible is that the particle is moving at ultrarelativistic velocities, i.e., we will set

$$
\dot{\xi} \equiv 1 - w^2 \, , \, w^2 \ll 1 \,. \tag{6}
$$

Insertion of Eq. (6) into Eq. (3) leads, after simple algebra, to the following reduced equation of motion for the variable w:

$$
\frac{dw}{dt} = \epsilon kw^3 \cos(k\xi) + \alpha w - \frac{\pi \gamma}{2\sqrt{2}} w^2 \sin(\omega t). \tag{7}
$$

In what follows below, we will concentrate on the case when the "particle" moves with a nearly constant velocity. In this case, the variable ξ in the argument of the cosine in the first term on the right-hand side of Eq. (7) may be replaced, in the lowest approximation, by

$$
\xi^{(0)} \equiv \overline{V}t + \xi_0, \qquad (8)
$$

where \overline{V} is the mean velocity (the same as in Eq. (5)), and ξ_0 is a constant phase shift between the particle's mean law of motion and the time-periodic driving force. Finally, introducing the renormalized time $\tau \equiv k\overline{V}t$, we obtain the lowest-order equation of motion in the form (with $\delta \equiv k \xi_0$)

$$
\frac{dw}{d\tau} = \frac{\epsilon}{\overline{V}} w^3 \cos(\tau + \delta) + \frac{\alpha}{k\overline{V}} w - \frac{\pi \gamma}{2\sqrt{2}k} w^2 \sin(\frac{\omega}{k\overline{V}} \tau). \tag{9}
$$

Now, we will make use of the fundamental assumption mentioned above, viz., that the driving and friction terms may be regarded as small in comparison with the first (potential) term on the right-hand side of Eq. (9). In the zero'th-order approximation, we drop the small terms, obtaining from Eq. (9)

$$
\frac{dw}{d\tau} = \frac{\epsilon}{\overline{V}} w^3 \cos(\tau + \delta). \tag{10}
$$

The exact solution of Eq. (10) is

$$
w = \sqrt{\frac{\Delta}{1 - \frac{2\epsilon\Delta}{\overline{V}}\sin(\tau + \delta)}}\,,\tag{11}
$$

where Δ is an arbitrary constant. According to Eq. (6), $w^2 \ll 1$, which implies that $\Delta \ll 1$. Moreover, the physically significant range for ϵ is $\epsilon \leq 1$; and finally, $\overline{V} \approx 1$. Consequently, the coefficient of the sine term in the denominator of Eq. (11) is small enough to replace this expression by its expansion,

$$
w = \sqrt{\Delta} \left[1 + \frac{\epsilon \Delta}{\overline{V}} \sin(\tau + \delta) \right]. \tag{12}
$$

Eqs. (6) and (12) describe the motion of a particle at the mean velocity $\overline{V} = 1 - \Delta$, with superimposed small oscillations of the velocity corresponding to the second term in the expression (12). Inserting this \overline{V} into the resonance condition of Eq. (5), we find the following relation between the driving frequency ω and the corresponding *resonant* value of the parameter Δ :

$$
\omega = k(1 - \Delta). \tag{13}
$$

Obviously, in a real physical situation the system chooses itself the resonant value of Δ corresponding to given ω . However, in the analytical consideration that follows below it will be more convenient to regard Δ as a given arbitrary parameter (small enough), and then to match the frequency to it according to Eq. (13).

A resonant drive can support progressive motion of a particle in a lossy periodic medium if the drive amplitude exceeds a certain minimum *(threshold)* value proportional to the friction coefficient [\[8\]](#page-12-0). In fact, the threshold value of the amplitude, regarded as a function of the driving frequency, is the basic characteristic of such ac-driven motion of the soliton in the damped system [\[8, 10\]](#page-12-0). In the present case, the amplitude of the driving force is the parameter γ . A convenient, approximate technique for calculating its threshold value, γ_{thr} , is to use the idea of harmonic balance [\[13\]](#page-12-0), *i.e.*, we insert the approximate solution given by Eq. (12) into the last two terms on the right-hand-side of Eq. (9), adjusting the coefficients in these terms so that the zero'th harmonic (and, in general, as many harmonics as possible) vanishes. The result of such a calculation (for the zero'th harmonic) is

$$
\gamma = \frac{4\sqrt{2}\alpha}{\pi\epsilon} \Delta^{-3/2} / |\cos(\delta)|. \tag{14}
$$

Recalling that δ is an arbitrarary phase shift, it is apparent that the threshold corresponds to $|\cos(\delta)| = 1$ in Eq. (14), *i.e.*,

$$
\gamma_{\rm thr} = \frac{4\sqrt{2}\alpha}{\pi\epsilon} \Delta^{-3/2} \,. \tag{15}
$$

This is our main analytical result. The proportionality of γ_{thr} to the dissipation constant α , as well as the inverse proportionality to ϵ , are obvious. A nonobvious feature of Eq. (15) is the power $-\frac{3}{2}$ $\frac{3}{2}$ of Δ (it is evident, of course, that this power must be negative). In the next section, comparing with results of direct numerical simulations of the equation of motion, Eq. (3), will show that this particular value of the power, as well as the numerical prefactor in Eq. (15), are quite reasonably accurate.

3 NUMERICAL RESULTS

In the numerical integration of Eq. (3), we fixed the driving frequency to $\omega = 1$. The justification for this step is provided by noting that ω can be scaled out of Eq. (3), *i.e.*, set to unity, by defining the following re-scaled variables: $t' \equiv \omega t$, $\xi' \equiv \omega \xi$ (which leaves $\dot{\xi}$ unchanged), $k' \equiv k/\omega$, $\alpha' \equiv \alpha/\omega$, $\gamma' \equiv \gamma/\omega$, and $\epsilon' \equiv \epsilon$. Such a re-scaling is valid provided that condition (4) continues to hold, but it obviously neglects any possible frequency-dependent resonance effects with the background oscillation (for the case when the driving frequency is close to the junction's plasma frequency, the resonance effects were analyzedin detail in [[14\]](#page-13-0)). Then, with Δ taken as an arbitrary parameter (small enough), we calculated the modulation wave number k from the resonant relation (13). The mode of simulations chosen was as follows: with all the parameters but the drive amplitude γ fixed, we reduced the value of γ quasi-adiabatically until the solution dropped out of synchronism with the driver; the smallest value of γ so obtained was taken as $\,\gamma_{\rm thr}$. The numerical results obtained are summarized in Figs. 1 and 2.

Figs. 1(a) and 1(b) show, respectively, the inverse and direct proportionality of $\gamma_{\rm thr}$ to the parameters ϵ and α . The analytical prediction of Eq. (15) is qualitatively well confirmed, even though there is a small quantitative discrepancy between analytical and numerical results. Fig. 2, instead, shows the numerical dependence on Δ . Here, we see that, for small enough values of Δ , the agreement between analytical and numerical results is quite good. The agreement gradually worsens as Δ increases, which is not surprising in view of the fact that $\Delta \ll 1$ is a fundamental assumption of the ultrarelativistic analysis leading to Eq. (15).

In the extreme nonrelativistic limit, *i.e.*, $\Delta \rightarrow 1$, which implies $\overline{V} \rightarrow 0$, Eq. (3) becomes, with trivial modifications, the equation for the damped, ac-driven, simple pendulum. The existence of a threshold value of the drive amplitude for phase-locked motion in this case was recently derived by Filatrella et al. [\[2](#page-12-0)] using an energy-balance argument. The solid curve in Fig. 2 shows the drive-amplitude threshold calculated for this case; as is evident from the figure, the numerical solution of Eq. (3) increasingly approaches this curve as $\Delta \to 1$.

In Ref.[[2](#page-12-0)], the numerical solutions of the "nonrelativistic" equation showed a gradual shrinkage, and eventual disappearance, with decreasing ω , of the basin of attraction in position-velocity space of the characteristic phase-locked motion. We have seen the same phenomenon, which, from Eq. (13), corresponds to the situation $\Delta \to 1$, in our present numerical solutions of Eq. (3). We suggest that this phenomenon might be attributable to the co-existence, in this region of parameter space, of chaotic trajectories in the underlying dynamics.

A crucial step in our analysis has been the reduction of the system description from the PDE of Eq. (1) to the ODE of Eq. (3). In order to check that the existence of ac-driven phase-locked states is not just an artifact of this reduction we have performed a very preliminary numerical study of Eq. (1), in which we have seen clear evidence, for several different parameter combinations, that the phenomenon is also present in the PDE[[15](#page-13-0)]. In these preliminary tests we applied periodic boundary conditions to Eq. (1), with the junction length chosen to contain one period of the spatial modulation, *i.e.*, $L = 2\pi/k$, with k given by Eq. (13). The PDE was discretized in space using a threepoint approximation for the second spatial derivative, and integrated in time using a fourth-order Runge-Kutta routine. Eq. (2), with $\xi = 0$ and $V = 0$, was used as the initial condition. In one such run, with $\alpha = 0.005$, $\epsilon = 1.0$, $\gamma = 0.53$, $\omega = 1.0$, $\Delta = 0.1$ and with a time step of 0.02 and a spatial grid of 100 points, the fluxon continued to propagate for at least 2500 normalized time units. Further detailed support for the very broad range of validity of the collective-coordinate (PDE-ODE) reduction in the periodically modulated

sine-Gordon equation (without loss or driving terms) has recently been presented by Sánchez *et al.* $|16|$.

4 CONCLUSIONS

In this work, we have demonstrated that a soliton in a periodically inhomogeneous continuum lossy medium can be driven at a nonzero mean velocity by an ac driving force. We have also found a good agreement between the analytical approximation based on harmonic balance and direct numerical simulations of the effective (ODE) equation of motion for the soliton. A noteworthy result is that the soliton can be easily driven in the "ultrarelativistic" region, but less easily at moderately relativistic and nonrelativistic values of the mean velocity. In this connection, one might reasonably ask, if the basin of attraction for ac-driven phase-locked states is small, whether or not such states can be of any potential physical interest, e, q , for practical applications. Our answer to this question is positive: the fact that an experimentalist might have to "prepare" his system, e.g., by initially applying a combination $dc + ac$ drive, and then gradually reducing the dc bias to zero, is not a serious impediment. What is important is that the basin of attraction be sufficiently large so that, once having arrived in an ac-driven phase-locked state (no matter how), the system be sufficiently stable against thermal noise and other perturbations, for a sufficiently long time to make measurements. The technique of cellto-cell mapping[[17, 18\]](#page-13-0), applied to Eq. (3), might provide a useful tool for obtaining some initial insight into this problem.

Finally, it is relevant to mention that in this work we have confined ourselves to consideration of the simplest resonance only. Higher-order resonances may also take place, in which $mk\overline{V} = n\omega$, with incommensurable integers m and n, cf. Eq. (5). It is natural to expect that the threshold amplitudes for the higher resonances will be considerably larger than for the fundamental resonance considered here.

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References

- [1] M. T. Levinsen, R. Y. Chiao, M. J. Feldman, and B. A. Tucker, Appl. Phys. Lett. 31 (1977) 776; R. L. Kautz, J. Appl. Phys. 52 (1981) 3528.
- [2] G. Filatrella, B. A. Malomed, and R. D. Parmentier, Phys. Lett. A 180 (1993) 346.
- [3] M. Salerno, M. R. Samuelsen, G. Filatrella, S. Pagano, and R. D. Parmentier, Phys. Rev. B 41 (1990) 6641; B. A. Malomed, Phys. Rev. B 41 (1990) 2037.
- [4] A. A. Golubov, I. L. Serpuchenko, and A. V. Ustinov, Zh. Eksp. Teor. Phys. 94 (1988) 296 [Sov. Phys. JETP 67 (1988) 385]; Phys. Lett. A 130 (1988) 107.
- [5] G. S. Mkrtchyan and V. V. Schmidt, Solid State Commun. 30 (1979) 791.
- [6] R. D. Parmentier, in: The New Superconducting Electronics, H. Weinstock and R. W. Ralston, eds. (Kluwer [NATO ASI Series, Vol. 251], Dordrecht, 1993), pp. 221- 248.
- [7] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. 61 (1989) 763.
- [8] B. A. Malomed, Phys. Rev. A 45 (1992) 4097.
- [9] L. Bonilla and B. A. Malomed, Phys. Rev. B 43 (1991) 11539.
- [10] T. Kuusela, J. Hietarinta, and B. A. Malomed, J. Phys. A: Math. Gen. 26, (1993) L21.
- [11] T. Kuusela, Phys. Lett. A 167 (1992) 54.
- [12] D. Cai and A. Sánchez, private communication.
- [13] N. Minorsky, Nonlinear Oscillations (Van Nostrand, New York, 1962), chap. 14.
- [14] B. A. Malomed, Physica D 27 (1987) 113.
- [15] G. Filatrella, unpublished.
- [16] A. Sánchez, A. R. Bishop, and F. Domínguez-Adame, "Kink stability, propagation, and length scale competition in the periodically modulated sine-Gordon equation", Phys. Rev. E (in press).
- [17] C. S. Hsu, Cell-to-Cell Mapping (Springer Verlag, Berlin, 1987).
- [18] N. F. Pedersen and A. Davidson, in: Stimulated Effects in Josephson Devices, M. Russo and G. Costabile, eds. (World Scientific, Singapore, 1990), pp. 227-238; M. P. Soerensen and N. F. Pedersen, ibid., pp. 292-308.

Figure Captions

Fig. 1. Dependence of the threshold value of the driving force, γ_{thr} , on the parameters: (a) ϵ (modulation coefficient), with $\alpha = 0.001$, $\omega = 1.0$, $\Delta = 0.1$; (b) α (dissipation coefficient), with $\epsilon = 1.0$, $\omega = 1.0$, $\Delta = 0.1$. Dash-dotted lines: prediction of Eq. (15); asterisks: numerical solutions of Eq. (3). Solid lines connecting the asterisks are a guide to the eye only.

Fig. 2. Dependence of the threshold value of the driving force, γ_{thr} , on the velocity parameter, Δ , with $\alpha = 0.001$, $\omega = 1.0$, $\epsilon = 0.5$. Dash-dotted line: prediction of Eq. (15); asterisks: numerical solution of Eq. (3); solid curve: analytical prediction of a nonrelativistic theory (see text). Solid lines connecting the asterisks are a guide to the eye only.

