

Explicit factorization of external coordinates in constrained Statistical Mechanics models

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Abstract

If a macromolecule is described by curvilinear coordinates or rigid constraints are imposed, the equilibrium probability density that must be sampled in Monte Carlo simulations includes the determinants of different mass-metric tensors. In this work, we explicitly write the determinant of the mass-metric tensor G and of the reduced mass-metric tensor g , for any molecule, general internal coordinates and arbitrary constraints, as a product of two functions; one depending only on the external coordinates that describe the overall translation and rotation of the system, and the other only on the internal coordinates. This work extends previous results in the literature, proving with full generality that one may integrate out the external coordinates and perform Monte Carlo simulations in the internal conformational space of macromolecules. In addition, we give a general mathematical argument showing that the factorization is a consequence of the symmetries of the metric tensors involved. Finally, the determinant of the mass-metric tensor G is computed explicitly in a set of curvilinear coordinates specially well-suited for general branched molecules.

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1 Introduction

Monte Carlo simulations are among the most useful tools for studying the behavior of macromolecules in thermal equilibrium [1–8]. Typically, the simulations are carried out in the *coordinate space*, i.e., the momenta are averaged out and Monte Carlo movements that only change the coordinates of the system are designed.

Also, the most interesting properties of macromolecules depend only on conformational transitions in the internal subspace of the whole coordinate space. The protein folding problem [9–12], the docking of ligands to proteins [13], or proteins to proteins [14], the prediction of Raman [15,16], IR [17,18], CD [19], VCD [20,21], NMR [22,23] spectra, etc. are tasks that require knowledge of the probability density in the *conformational space* only, i.e., having averaged out the external coordinates that describe overall translations and rotations of the system.

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If Cartesian coordinates are used, the integration over the momenta produces a constant factor (which depends on the temperature T but does not depend on the coordinates) and the marginal probability density in the coordinate space resembles the common Boltzmann weight but using the potential energy $V(x^\mu)$ instead of the whole energy:

$$p_c(x^\mu) = \frac{\exp[-\beta V(x^\mu)]}{\int dx^\nu \exp[-\beta V(x^\nu)]} . \quad (1.1)$$

Typically the potential energy does not change under global translations and rotations of the system. In addition, as we have already mentioned, one is normally not interested in averages of observables that depend on these degrees of freedom. Hence, it would be convenient to average them out from eq. (1.1). However, this cannot be done in Cartesian coordinates: one must use a set of coordinates adapted to overall translations and rotations.

In the simulation of macromolecules, it is customary [8, 24–30] to define a set of curvilinear coordinates q^μ in which the first six ones, denoted by q^A , are called *external coordinates* and parametrize the system overall position, specifying the position of a selected point (normally an atom), and rotation, via three Euler angles (see sec. 2). The remaining $3n - 6$ coordinates (where n is the number of mass points or *atoms*) are called *internal coordinates* and will be denoted herein by q^a .

This change of coordinates modifies the mass-metric tensor in the kinetic energy. Thus, when the momenta are averaged out and the marginal probability density in the whole coordinate space is considered, the square root of the determinant of the mass-metric tensor (which now does depend on the coordinates) shows up:

$$p_w(q^\mu) = \frac{\det^{\frac{1}{2}} G(q^A, q^a) \exp[-\beta V(q^a)]}{\int dq^B dq^b \det^{\frac{1}{2}} G(q^B, q^b) \exp[-\beta V(q^b)]} . \quad (1.2)$$

More interestingly, if holonomic constraints are imposed on the system (the so-called *classical rigid* model [31–33]), the reduced mass-metric tensor on the constrained hypersurface appears in the kinetic energy. Hence, when the momenta are integrated out from the joint probability density in the phase space, the square root of its determinant occurs:

$$p_r(q^u) = \frac{\det^{\frac{1}{2}} g(q^A, q^i) \exp[-\beta V_\Sigma(q^i)]}{\int dq^B dq^j \det^{\frac{1}{2}} g(q^B, q^j) \exp[-\beta V_\Sigma(q^j)]} , \quad (1.3)$$

where V_Σ stands for the potential energy in the constrained hypersurface Σ , $q^u \equiv (q^A, q^i)$ denotes the *soft coordinates*, among which the external ones q^A are included, and q^i denotes the soft internal coordinates.

If, on the other hand, the constraints are imposed via a steep potential that energetically penalizes the conformations that leave the constrained hypersurface (the so-called *classical stiff* model [30–33]), the probability density is the same as in eq. (1.2) except for the determinant of the Hessian matrix \mathcal{H} of the constraining potential that appears when the *hard coordinates* are averaged out and for the fact that all the functions are evaluated on the constrained hypersurface and, consequently, depend only on the soft coordinates q^u :

$$p_s(q^u) = \frac{\det^{\frac{1}{2}} G(q^A, q^i) \det^{-\frac{1}{2}} \mathcal{H}(q^i) \exp[-\beta V_\Sigma(q^i)]}{\int dq^B dq^j \det^{\frac{1}{2}} G(q^B, q^j) \det^{-\frac{1}{2}} \mathcal{H}(q^j) \exp[-\beta V_\Sigma(q^j)]} . \quad (1.4)$$

Finally, the Fixman’s compensating potential [31–34], denoted by V_F and which is customarily used to reproduce the stiff equilibrium distribution using rigid molecular dynamics simulations [8, 31, 35], is also expressed as a function of these determinants:

$$V_F(q^u) := \frac{RT}{2} \ln \left[\frac{\det^{\frac{1}{2}} g(q^A, q^i) \det^{\frac{1}{2}} \mathcal{H}(q^i)}{\det^{\frac{1}{2}} G(q^A, q^i)} \right]. \quad (1.5)$$

One should note that the fields of application of the different mass-metric tensor determinants are distinct. The Fixman’s compensating potential above is only meant to allow that the stiff distribution in eq. (1.4) be sampled in rigid Molecular Dynamics simulations [33, 35–45]. and *it should never be included in Monte Carlo simulations*. On the other hand, if one chooses as his physical description the whole space, the rigid or the stiff model, the probability densities that must be sampled are the ones in eqs. (1.2), (1.3) and (1.4), respectively [30–34, 44, 46]. Due to the averaging out of the momenta in these expressions, the determinants of either G or g show up, hence, if Monte Carlo simulations of these models are to be performed, these corrections (which are related to but different from the Fixman’s compensating potential) should be included or, otherwise, shown to be negligible. The discussion found in the literature about the necessity of including these terms [47] is based on different simplifications, approximations and misconceptions. On one side, since Gō and Scheraga showed some decades ago that the determinant of G , for a serial polymer with constant bond angles and bond lengths, does not depend on the conformation of the molecule [30], it is customarily neglected in the literature. However, this must be understood as an approximation, since, as some authors have recognized [30, 39, 48, 49], the constrained values of the hard coordinates depend on the soft ones even in the case of simple force-fields, due to the long-range energy terms and, in such a case, $\det G$ does depend on the q^i , rendering pertinent its inclusion in the basic equations. Moreover, the determinant of the Hessian \mathcal{H} of the constraining part of the potential is also assumed to be independent of conformation by most authors [30, 33, 50]. This is also an approximation, due to the same reasons presented above, and should be assessed in each case, however, if these two approximations are made (the neglect of $\det G$ and of $\det \mathcal{H}$), the Fixman’s compensating potential in eq. (1.5) depends only on $\det g$, just as the correcting term in the rigid case. This is one of the sources of confusion and the reason that the relevance of mass-metric effects is associated only with the conformational dependence of the determinant of the reduced mass-metric tensor g [47]. One should also note that most authors accept that the *correct* constrained model is the stiff one [33, 34, 38, 42, 47, 50, 51], and, therefore, no works are written in which the rigid probability density in eq. (1.3) is sampled. In our opinion, the question whether the rigid or the stiff model should be used to approximate the real quantum mechanical statistics of an arbitrary organic molecule has not been satisfactorily answered yet. For discussions about the topic, see references [30, 32, 42, 44, 46, 49, 52]. In this work, we adopt the cautious position that any of the two models may be useful in certain cases or for certain purposes and we study them both on equal footing. All these approximations together are the reason of the fact that no determinants are included in Monte Carlo simulations and that the Fixman’s compensating potential is erroneously regarded as playing a role outside Molecular Dynamics simulations.

In the three physical models, given by eqs. (1.2), (1.3) and (1.4), and in the Fixman’s potential written above, neither the potential energy nor the Hessian matrix of the constraining potential depend on the external coordinates. Therefore, it would be very convenient to integrate them out in order to obtain a simpler probability density depending only on the internal coordinates. Such an improvement may render the coding of computer applications and the visualization of molecules easier, since all the movements in the molecule may be performed fixing an atom in space and keeping the orientation

of the system with respect to a set of axes fixed in space constant [53]. Moreover, the insight gained and the usefulness of the calculations herein in further analytical studies constitute additional benefits. Regarding Monte Carlo simulations, it is not clear that much computational effort will be saved for macromolecules, since the gain expected when reducing the degrees of freedom from $M + 6$ to M (where M is the number of soft internal coordinates) is only appreciable if M is small and the difficulties arising from the use of curvilinear coordinates may well be more important. However, in cases where the use of curvilinear coordinates is a must, such as the simulation of constrained systems, the calculations in this work allow to save a time (which will depend on the size of the system) that, otherwise, would be wasted in movements of the external coordinates. Finally, if the molecule treated is small, as it is common in ab initio Quantum Mechanical calculations in model peptides [54–58], the relevance of omitting the externals may be considerable. In a work recently done in our group [31], in which the model dipeptide HCO-L-Ala-NH₂ is studied, the soft internals are two: the Ramachandran angles ϕ and ψ ; hence, $M = 2$ and the formulae in this work have permitted to reduce the number of degrees of freedom from 8 to 2. This situation is very common in the literature [54–58].

Now, for the resulting expressions to be manageable, the determinant of the mass-metric tensor G , in p_w or p_s , and the determinant of g , in the rigid case, should *factorize* as a product of a function that depends only on the external coordinates and another function that depends only on the internal ones. Then the function depending on the external coordinates, could be integrated out in the probability densities p_w , p_r and p_s or taken out of the logarithm in V_F ¹.

For some simple examples, it has already been proven in the literature that this factorization actually happens. In ref. 30, the determinant of G is shown to factorize for a serial polymer in a particular set of curvilinear coordinates. In ref. 47, the determinant of g is shown to factorize for the same system, in similar coordinates, with frozen bond lengths and bond angles.

In this work, we *generalize* these results, showing that they hold in *arbitrary* internal coordinates (for general branched molecules) and with *arbitrary* constraints. Perhaps more importantly, we provide explicit expression for the functions involved in the factorization. It is worth remarking that, although the calculations herein have been performed thinking in macromolecules as target system, they are completely general and applicable to any classical system composed by discrete mass points.

In sec. 2, we present the notation and conventions that will be used throughout the article. In sec. 3, we explicitly factorize the determinant of the reduced mass-metric tensor g as a product of a function that depends only on the external coordinates and another function that depends on arbitrary internal coordinates; no specific form is assumed for the constraints. In sec. 4, we perform the analogous calculations for the determinant of the mass-metric tensor G and sec. 5 is devoted to the conclusions. In appendix A, the general mathematical argument underlying these results is given. Finally, in appendix B, the determinant of the mass-metric tensor G is computed in the set of curvilinear coordinates introduced in ref. 24, which turn out to be convenient for dealing with general branched molecules. Moreover, we show that the classical formula for serial polymers [30] is actually valid for any macromolecule.

2 General set-up and definitions

The system under scrutiny will be a set of n mass points termed *atoms*. This section is devoted to introduce certain notational conventions that will be used extensively in the

¹What really happens, (see secs. 3 and 4) is that the factor that depends on the external coordinates is the same for $\det G$ and $\det g$. Hence, it divides out in eq. (1.5) (see sec. 5).

rest of the paper.

- The superindex T indicates matrix transposition. By \vec{a}^T we shall understand the row vector (a^1, a^2, a^3) .
- The Cartesian coordinates of the atom α in a set of axes fixed in space are denoted by \vec{x}_α . The subscript α runs from 1 to n .
- The curvilinear coordinates suitable to integrate out the external degrees of freedom will be denoted by q^μ , $\mu = 1, \dots, 3n$. We shall often use $N := 3n$ for the total number of degrees of freedom.
- We choose the coordinates q^μ so that the first six are *external coordinates*. They are denoted by q^A and their ordering is $q^A \equiv (X, Y, Z, \phi, \theta, \psi)$. The first three ones, $\vec{X}^T := (X, Y, Z)$, describe the overall position of the system. The three angles (ϕ, θ, ψ) are related to its overall orientation. More concretely, they give the orientation of a frame fixed in the system with respect to the frame fixed in space.
- To define the set of axes *fixed in the system*, we select three atoms (denoted by 1, 2 and 3) in such a way that \vec{X} is the position of atom 1 (i.e., $\vec{x}_1 = \vec{X}$). The orientation of the fixed axes (x', y', z') is chosen such that atom 2 lies in the positive half of the z' -axis and atom 3 is contained in the (x', z') -plane, in the positive half of the x' -axis (see fig. 1). The position of atom α in these axes is denoted by \vec{x}'_α .

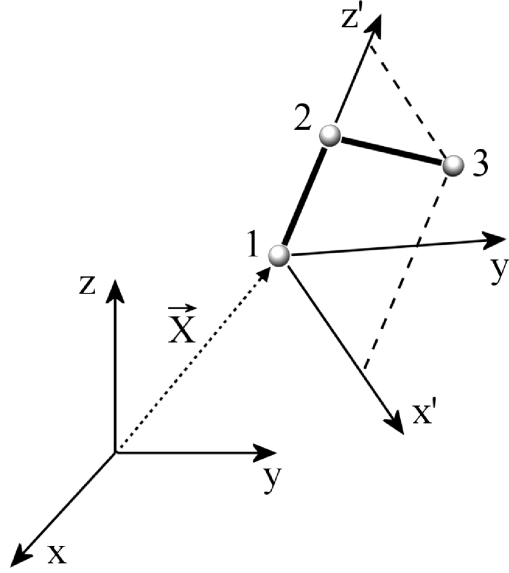


Figure 1: Definition of the axes fixed in the system.

- Let $E(\phi, \theta, \psi)$ be the Euler rotation matrix (in the ZYZ convention) that takes a vector of primed components \vec{a}' to the frame fixed in space, i.e., $\vec{a} = E(\phi, \theta, \psi) \vec{a}'$. Its explicit expression is the following:

$$E(\phi, \theta, \psi) = \underbrace{\begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Phi(\phi)} \underbrace{\begin{pmatrix} -\cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & -\cos \theta \end{pmatrix}}_{\Theta(\theta)} \underbrace{\begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Psi(\psi)}. \quad (2.1)$$

The unusual minus signs of the cosines in the diagonal of matrix $\Theta(\theta)$ come from the fact that, due to frequent biochemical conventions, the rotation with respect to the y -axis is of angle $\tilde{\theta} := \pi - \theta$.

- The coordinates q^μ are split into (q^A, q^a) , $a = 7, \dots, N$. The coordinates q^a are said *internal coordinates* and determine the positions of the atoms in the frame fixed in the system. The transformation from the Cartesian coordinates \vec{x}_α to the curvilinear coordinates q^μ may be written as follows:

$$\vec{x}_\alpha = \vec{X} + E(\phi, \theta, \psi) \vec{x}'_\alpha(q^a) \quad \alpha = 1, \dots, n. \quad (2.2)$$

- The coordinates q^a parameterize what we shall call the *internal subspace*, denoted by \mathcal{I} . Assume that L independent constraints are imposed on \mathcal{I} , so that only points on a hypersurface $\Sigma \subset \mathcal{I}$ of dimension $M := N - L - 6$ are allowed. Then, we choose a splitting $q^a \equiv (q^i, q^I)$, with $i = 7, \dots, M + 6$ and $I = M + 7, \dots, N$, where q^i (the *internal soft coordinates*) parameterize Σ , and q^I (the *hard coordinates*) are functions of the soft coordinates:

$$q^I = f^I(q^i) \quad I = M + 7, \dots, N. \quad (2.3)$$

If these constraints are used, together with eq. (2.2), the Cartesian position of any atom may be parameterized with the set of *all soft coordinates*, denoted by $q^u \equiv (q^A, q^i)$, with $u = 1, \dots, M + 6$, as follows:

$$\vec{x}_\alpha = \vec{X} + E(\phi, \theta, \psi) \vec{x}'_\alpha(q^i, f^I(q^i)) \quad \alpha = 1, \dots, n. \quad (2.4)$$

- In table 1, a summary of the indices used is given.

Indices	Range	Number	Description
$\alpha, \beta, \gamma, \dots$	$1, \dots, n$	n	Atoms
p, q, r, s, \dots	$1, 2, 3$	3	Components of trivectors
μ, ν, ρ, \dots	$1, \dots, N$	$N = 3n$	All coordinates
A, B, C, \dots	$1, \dots, 6$	6	External coordinates
a, b, c, \dots	$7, \dots, N$	$N - 6$	Internal coordinates
i, j, k, \dots	$7, \dots, M + 6$	M	Soft internal coordinates
I, J, K, \dots	$M + 7, \dots, N$	$L = N - M - 6$	Hard internal coordinates
u, v, w, \dots	$1, \dots, M + 6$	$M + 6$	All soft coordinates

Table 1: Definition of the indices used.

3 Constrained case

The *reduced mass-metric tensor*, in the constrained hypersurface Σ plus the external subspace spanned by the q^A , may be written as follows:

$$g_{vw}(q^u) := \sum_{\mu=1}^N \frac{\partial x^\mu(q^u)}{\partial q^v} m_\mu \frac{\partial x^\mu(q^u)}{\partial q^w}. \quad (3.1)$$

In matrix notation, this is written as

$$g = J_c^T M J_c, \quad (3.2)$$

where c stands for *constrained* and M is the diagonal $N \times N$ mass matrix given by

$$M := \begin{pmatrix} m_1^{(3)} & & 0 \\ & \ddots & \\ 0 & & m_n^{(3)} \end{pmatrix}, \quad \text{with } m_\alpha^{(3)} := m_\alpha \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I^{(3)}}. \quad (3.3)$$

Using eq. (2.4) and noting that the derivatives with respect to the externals, q^A , only affect the \vec{X} vector and the Euler rotation matrix E , while differentiation with respect to soft internals, q^i , only act on the \vec{x}'_α , we have that J_c is the $N \times (M + 6)$ matrix

$$J_c = \left(\begin{array}{ccc|ccc} I^{(3)} & & 0 & & & 0 \\ I^{(3)} & \frac{\partial E}{\partial \phi} \vec{x}'_2 & \frac{\partial E}{\partial \theta} \vec{x}'_2 & \frac{\partial E}{\partial \psi} \vec{x}'_2 & \cdots & E \frac{\partial \vec{x}'_2}{\partial q^j} \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ I^{(3)} & \frac{\partial E}{\partial \phi} \vec{x}'_\alpha & \frac{\partial E}{\partial \theta} \vec{x}'_\alpha & \frac{\partial E}{\partial \psi} \vec{x}'_\alpha & \cdots & E \frac{\partial \vec{x}'_\alpha}{\partial q^j} \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{array} \right). \quad (3.4)$$

If we perform the matrix multiplications in eq. (3.2), we obtain

$$g = \left(\begin{array}{cc|ccc} I^{(3)} \sum_\alpha m_\alpha & \sum_\alpha m_\alpha \partial E_\alpha & \cdots & \sum_\alpha m_\alpha E \frac{\partial \vec{x}'_\alpha}{\partial q^j} & \cdots \\ \sum_\alpha m_\alpha \partial E_\alpha^T & \sum_\alpha m_\alpha \partial E_\alpha^T \partial E_\alpha & \cdots & \sum_\alpha m_\alpha \partial E_\alpha^T E \frac{\partial \vec{x}'_\alpha}{\partial q^j} \cdots & \\ \vdots & \vdots & & \vdots & \\ \sum_\alpha m_\alpha \frac{\partial \vec{x}'_\alpha{}^T}{\partial q^i} E^T & \sum_\alpha m_\alpha \frac{\partial \vec{x}'_\alpha{}^T}{\partial q^i} E^T \partial E_\alpha & \cdots & \sum_\alpha m_\alpha \frac{\partial \vec{x}'_\alpha{}^T}{\partial q^i} \frac{\partial \vec{x}'_\alpha}{\partial q^j} \cdots & \\ \vdots & \vdots & & \vdots & \end{array} \right). \quad (3.5)$$

All the sums in α can be understood as ranging from 1 to n if we note that $\vec{x}'_1 = \vec{0}$. In the bottom right block, the fact that E is an orthogonal matrix (i.e., that $E^T E = I^{(3)}$) has been used, and we have defined the 3×3 block as

$$\partial E_\alpha := \left(\frac{\partial E}{\partial \phi} \vec{x}'_\alpha \quad \frac{\partial E}{\partial \theta} \vec{x}'_\alpha \quad \frac{\partial E}{\partial \psi} \vec{x}'_\alpha \right). \quad (3.6)$$

We can write g as

$$g := \begin{pmatrix} E & 0 \\ 0 & I^{(M+3)} \end{pmatrix} g_1 \begin{pmatrix} E^T & 0 \\ 0 & I^{(M+3)} \end{pmatrix}, \quad (3.7)$$

where $I^{(M+3)}$ is the $(M + 3) \times (M + 3)$ identity matrix and g_1 is defined as

$$g_1 := \left(\begin{array}{cc|ccc} I^{(3)} \sum_{\alpha} m_{\alpha} & \sum_{\alpha} m_{\alpha} E^T \partial E_{\alpha} & \cdots & \sum_{\alpha} m_{\alpha} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \sum_{\alpha} m_{\alpha} \partial E_{\alpha}^T E & \sum_{\alpha} m_{\alpha} \partial E_{\alpha}^T E E^T \partial E_{\alpha} & \cdots & \sum_{\alpha} m_{\alpha} \partial E_{\alpha}^T E \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & & \vdots & \\ \sum_{\alpha} m_{\alpha} \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} & \sum_{\alpha} m_{\alpha} \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} E^T \partial E_{\alpha} & \cdots & \sum_{\alpha} m_{\alpha} \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \right). \quad (3.8)$$

Note that $I^{(3)} = EE^T$ has been introduced in the bottom right 3×3 submatrix of the top left block.

Next, we introduce some simplifying notation for the matrices $E^T \partial E_{\alpha}$:

$$E^T \partial E_{\alpha} = \left(E^T \frac{\partial E}{\partial \phi} \bar{x}'_{\alpha} \quad E^T \frac{\partial E}{\partial \theta} \bar{x}'_{\alpha} \quad E^T \frac{\partial E}{\partial \psi} \bar{x}'_{\alpha} \right) =: \left(\bar{y}_{\alpha}^1 \quad \bar{y}_{\alpha}^2 \quad \bar{y}_{\alpha}^3 \right). \quad (3.9)$$

Defining

$$g_1^{\alpha} := \left(\begin{array}{cccc|ccc} I^{(3)} & \bar{y}_{\alpha}^1 & \bar{y}_{\alpha}^2 & \bar{y}_{\alpha}^3 & \cdots & \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \bar{y}_{\alpha}^{1T} & \bar{y}_{\alpha}^{1T} \bar{y}_{\alpha}^1 & \bar{y}_{\alpha}^{1T} \bar{y}_{\alpha}^2 & \bar{y}_{\alpha}^{1T} \bar{y}_{\alpha}^3 & \cdots & \bar{y}_{\alpha}^{1T} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \bar{y}_{\alpha}^{2T} & \bar{y}_{\alpha}^{2T} \bar{y}_{\alpha}^1 & \bar{y}_{\alpha}^{2T} \bar{y}_{\alpha}^2 & \bar{y}_{\alpha}^{2T} \bar{y}_{\alpha}^3 & \cdots & \bar{y}_{\alpha}^{2T} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \bar{y}_{\alpha}^{3T} & \bar{y}_{\alpha}^{3T} \bar{y}_{\alpha}^1 & \bar{y}_{\alpha}^{3T} \bar{y}_{\alpha}^2 & \bar{y}_{\alpha}^{3T} \bar{y}_{\alpha}^3 & \cdots & \bar{y}_{\alpha}^{3T} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \bar{y}_{\alpha}^1 & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \bar{y}_{\alpha}^2 & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \bar{y}_{\alpha}^3 & \cdots & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{array} \right), \quad (3.10)$$

we have that

$$g_1 = \sum_{\alpha} m_{\alpha} g_1^{\alpha}. \quad (3.11)$$

Now, the vectors \bar{y}_{α}^p may be extracted from g_1^{α} as follows:

$$g_1^{\alpha} = Y_{\alpha}^T \left(\begin{array}{cc|ccc} I^{(3)} & I^{(3)} & \cdots & \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ I^{(3)} & I^{(3)} & \cdots & \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & & \vdots & \\ \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} & \cdots & \frac{\partial \bar{x}'_{\alpha}{}^T}{\partial q^i} \frac{\partial \bar{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \right) Y_{\alpha}, \quad (3.12)$$

where

$$Y_\alpha := \begin{pmatrix} I^{(3)} & 0 & 0 \\ 0 & \vec{y}_\alpha^1 \vec{y}_\alpha^2 \vec{y}_\alpha^3 & 0 \\ 0 & 0 & I^{(M)} \end{pmatrix}, \quad (3.13)$$

and the central matrix in eq. (3.12) only depends on the soft internal coordinates. After some lengthy calculations, one shows that

$$\begin{pmatrix} \vec{y}_\alpha^1 & \vec{y}_\alpha^2 & \vec{y}_\alpha^3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -x_\alpha'^3 & x_\alpha'^2 \\ x_\alpha'^3 & 0 & -x_\alpha'^1 \\ -x_\alpha'^2 & x_\alpha'^1 & 0 \end{pmatrix}}_{v(\vec{x}'_\alpha)} \underbrace{\begin{pmatrix} \sin \theta \cos \psi & \sin \psi & 0 \\ -\sin \theta \sin \psi & \cos \psi & 0 \\ \cos \theta & 0 & -1 \end{pmatrix}}_{W(\theta, \psi)}. \quad (3.14)$$

Thus, the matrix Y_α in eq. (3.13) may be written as

$$Y_\alpha = \begin{pmatrix} I^{(3)} & 0 & 0 \\ 0 & v(\vec{x}'_\alpha) & 0 \\ 0 & 0 & I^{(M)} \end{pmatrix} \begin{pmatrix} I^{(3)} & 0 & 0 \\ 0 & W(\theta, \psi) & 0 \\ 0 & 0 & I^{(M)} \end{pmatrix}. \quad (3.15)$$

If we take this expression to eq. (3.12) and use that, for any pair of vectors \vec{a} and \vec{b} , $\vec{a}^T v(\vec{b}) = (\vec{a} \times \vec{b})^T$ and $v^T(\vec{b}) \vec{a} = \vec{a} \times \vec{b}$, where \times denotes the usual vector cross product, we may rewrite eq. (3.12) as follows:

$$g_1^\alpha = \begin{pmatrix} I^{(3)} & 0 & 0 \\ 0 & W^T(\theta, \psi) & 0 \\ 0 & 0 & I^{(M)} \end{pmatrix} g_2^\alpha \begin{pmatrix} I^{(3)} & 0 & 0 \\ 0 & W(\theta, \psi) & 0 \\ 0 & 0 & I^{(M)} \end{pmatrix}, \quad (3.16)$$

where g_2^α is defined as

$$g_2^\alpha = \left(\begin{array}{cc|ccc} I^{(3)} & v(\vec{x}'_\alpha) & \cdots & \frac{\partial \vec{x}'_\alpha}{\partial q^j} & \cdots \\ v^T(\vec{x}'_\alpha) & v^T(\vec{x}'_\alpha)v(\vec{x}'_\alpha) & \cdots & \frac{\partial \vec{x}'_\alpha}{\partial q^j} \times \vec{x}'_\alpha & \cdots \\ \hline \vdots & \vdots & & \vdots & \\ \frac{\partial \vec{x}'_\alpha{}^T}{\partial q^i} & \left(\frac{\partial \vec{x}'_\alpha}{\partial q^i} \times \vec{x}'_\alpha \right)^T & \cdots & \frac{\partial \vec{x}'_\alpha{}^T}{\partial q^i} \frac{\partial \vec{x}'_\alpha}{\partial q^j} & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \right). \quad (3.17)$$

At this point, we insert eq. (3.16) in eq. (3.11) and take the W matrices out of the sum. Since $\det W(\theta, \psi) = \det W^T(\theta, \psi) = -\sin \theta$, we obtain

$$\det g_1 = \sin^2 \theta \det \underbrace{\left(\sum_\alpha m_\alpha g_2^\alpha \right)}_{g_2}. \quad (3.18)$$

Recalling that $\det E = \det E^T = 1$, from eq. (3.7), we have that

$$\det g(q^A, q^i) = \det g_1(q^A, q^i) = \sin^2 \theta \det g_2(q^i), \quad (3.19)$$

and the factorization of the external coordinates has been finally accomplished, since g_2 is the following matrix, which depends only on the soft internal coordinates q^i :

$$g_2 = \left(\begin{array}{cc|ccc} m_{tot} I^{(3)} & m_{tot} v(\vec{R}) & \cdots & m_{tot} \frac{\partial \vec{R}}{\partial q^j} & \cdots \\ m_{tot} v^T(\vec{R}) & \mathcal{J} & \cdots & \sum_{\alpha} m_{\alpha} \frac{\partial \vec{x}'_{\alpha}}{\partial q^j} \times \vec{x}'_{\alpha} & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{tot} \frac{\partial \vec{R}}{\partial q^i} & \sum_{\alpha} m_{\alpha} \left(\frac{\partial \vec{x}'_{\alpha}}{\partial q^i} \times \vec{x}'_{\alpha} \right)^T & \cdots & \sum_{\alpha} m_{\alpha} \frac{\partial \vec{x}'_{\alpha}{}^T}{\partial q^i} \frac{\partial \vec{x}'_{\alpha}}{\partial q^j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right), \quad (3.20)$$

where we have defined the *total mass* of the system $m_{tot} := \sum_{\alpha} m_{\alpha}$, the position of the *center of mass* of the system in the primed reference frame $\vec{R} := m_{tot}^{-1} \sum_{\alpha} m_{\alpha} \vec{x}'_{\alpha}$ and the *inertia tensor* of the system, also in the primed reference frame:

$$\mathcal{J} := \begin{pmatrix} \sum_{\alpha} m_{\alpha} ((x'_{\alpha}{}^2)^2 + (x'_{\alpha}{}^3)^2) & -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^1 x'_{\alpha}{}^2 & -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^1 x'_{\alpha}{}^3 \\ -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^1 x'_{\alpha}{}^2 & \sum_{\alpha} m_{\alpha} ((x'_{\alpha}{}^1)^2 + (x'_{\alpha}{}^3)^2) & -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^2 x'_{\alpha}{}^3 \\ -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^1 x'_{\alpha}{}^3 & -\sum_{\alpha} m_{\alpha} x'_{\alpha}{}^2 x'_{\alpha}{}^3 & \sum_{\alpha} m_{\alpha} ((x'_{\alpha}{}^1)^2 + (x'_{\alpha}{}^2)^2) \end{pmatrix}. \quad (3.21)$$

4 Unconstrained case

If no constraints are assumed and the system lives in the whole internal space \mathcal{I} plus the external subspace spanned by the q^A , the Cartesian coordinates of the n atoms must be expressed using eq. (2.2), instead of eq. (2.4).

We now wish to calculate the determinant of the *whole-space* mass-metric tensor in the coordinates q^{μ} :

$$G_{\nu\rho}(q^{\mu}) := \sum_{\sigma=1}^N \frac{\partial x^{\sigma}(q^{\mu})}{\partial q^{\nu}} m_{\sigma} \frac{\partial x^{\sigma}(q^{\mu})}{\partial q^{\rho}}, \quad (4.1)$$

which, in matrix form, reads

$$G = J^T M J. \quad (4.2)$$

The only difference with eq. (3.2) is that, instead of the rectangular matrix J_c (see eq. (3.4)), in the above expression the full *Jacobian matrix* of the change of coordinates from Cartesian to curvilinear coordinates appears:

$$J_{\rho}^{\sigma}(q^{\mu}) := \frac{\partial x^{\sigma}(q^{\mu})}{\partial q^{\rho}}. \quad (4.3)$$

Obviously, one can deduce the factorization of $\det G$ as a particular case of the results of sec. 3 with $L = 0$, so that the indices i, j now run over all internal coordinates q^a . Explicitly,

$$\det G(q^A, q^a) = \sin^2 \theta \det G_2(q^a), \quad (4.4)$$

with

$$G_2 := \left(\begin{array}{cc|ccc} m_{tot} I^{(3)} & m_{tot} v(\vec{R}) & \cdots & m_{tot} \frac{\partial \vec{R}}{\partial q^b} & \cdots \\ m_{tot} v^T(\vec{R}) & \mathcal{J} & \cdots & \Sigma_{\alpha} m_{\alpha} \frac{\partial \vec{x}'_{\alpha}}{\partial q^b} \times \vec{x}'_{\alpha} & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{tot} \frac{\partial \vec{R}}{\partial q^a} & \Sigma_{\alpha} m_{\alpha} \left(\frac{\partial \vec{x}'_{\alpha}}{\partial q^a} \times \vec{x}'_{\alpha} \right)^T & \cdots & \Sigma_{\alpha} m_{\alpha} \frac{\partial \vec{x}'_{\alpha}{}^T}{\partial q^a} \frac{\partial \vec{x}'_{\alpha}}{\partial q^b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right). \quad (4.5)$$

However, in this section we would like to benefit from the special structure of eq. (4.2), where, differently from the constrained case, only $N \times N$ matrices occur, and find an expression simpler than eq. (4.4).

If we take determinants on both sides of eq. (4.2), we obtain

$$\det G = \left(\prod_{\alpha=1}^n m_{\alpha}^3 \right) \det^2 J. \quad (4.6)$$

Similarly to eq. (3.4), J may be written as follows:

$$J = \left(\begin{array}{cccc|ccc} I^{(3)} & 0 & & & 0 & & \\ \hline I^{(3)} & \frac{\partial E}{\partial \phi} \vec{x}'_2 & \frac{\partial E}{\partial \theta} \vec{x}'_2 & \frac{\partial E}{\partial \psi} \vec{x}'_2 & \cdots & E \frac{\partial \vec{x}'_2}{\partial q^b} & \cdots \\ I^{(3)} & \frac{\partial E}{\partial \phi} \vec{x}'_3 & \frac{\partial E}{\partial \theta} \vec{x}'_3 & \frac{\partial E}{\partial \psi} \vec{x}'_3 & \cdots & E \frac{\partial \vec{x}'_3}{\partial q^b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I^{(3)} & \frac{\partial E}{\partial \phi} \vec{x}'_{\alpha} & \frac{\partial E}{\partial \theta} \vec{x}'_{\alpha} & \frac{\partial E}{\partial \psi} \vec{x}'_{\alpha} & \cdots & E \frac{\partial \vec{x}'_{\alpha}}{\partial q^b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right). \quad (4.7)$$

Now, the following identity is useful:

$$J = \begin{pmatrix} I^{(3)} & & 0 \\ & E & \\ & & \ddots \\ 0 & & & E \end{pmatrix} J_1, \quad (4.8)$$

where we have defined

$$J_1 := \left(\begin{array}{cccc|ccc} I^{(3)} & 0 & & & 0 & & \\ \hline I^{(3)} & E^T \frac{\partial E}{\partial \phi} \vec{x}'_2 & E^T \frac{\partial E}{\partial \theta} \vec{x}'_2 & E^T \frac{\partial E}{\partial \psi} \vec{x}'_2 & \cdots & \frac{\partial \vec{x}'_2}{\partial q^b} & \cdots \\ I^{(3)} & E^T \frac{\partial E}{\partial \phi} \vec{x}'_3 & E^T \frac{\partial E}{\partial \theta} \vec{x}'_3 & E^T \frac{\partial E}{\partial \psi} \vec{x}'_3 & \cdots & \frac{\partial \vec{x}'_3}{\partial q^b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I^{(3)} & E^T \frac{\partial E}{\partial \phi} \vec{x}'_{\alpha} & E^T \frac{\partial E}{\partial \theta} \vec{x}'_{\alpha} & E^T \frac{\partial E}{\partial \psi} \vec{x}'_{\alpha} & \cdots & \frac{\partial \vec{x}'_{\alpha}}{\partial q^b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right), \quad (4.9)$$

and we have that only the determinant of J_1 needs to be computed, since $\det J = \det^{n-1} E \det J_1 = \det J_1$.

Next, we note that, according to the definition of the primed reference frame in sec. 2, some of the components of the vectors \vec{x}'_2 and \vec{x}'_3 are zero, namely, we have that

$$\vec{x}'_2 = \begin{pmatrix} 0 \\ 0 \\ x_2'^3 \end{pmatrix} \quad \text{and} \quad \vec{x}'_3 = \begin{pmatrix} x_3'^1 \\ 0 \\ x_3'^3 \end{pmatrix}. \quad (4.10)$$

Hence, the derivatives with respect to q^b of the zero components are also zero, rendering three zero rows in the bottom right block of eq. (4.9). Performing two row permutations so that the zero rows are the top-most ones, we obtain a matrix J_2 whose determinant is the same as the one of J_1 :

$$J_2 = \begin{pmatrix} I^{(3)} & 0 \\ \mathcal{X} & J_2^\mathcal{E} \\ & & J_2^\mathcal{I} \end{pmatrix}, \quad (4.11)$$

where the blocks in the diagonal have been defined as

$$J_2^\mathcal{E} = \begin{pmatrix} \left(E^T \frac{\partial E}{\partial \phi} \vec{x}'_2 \right)^1 & \left(E^T \frac{\partial E}{\partial \theta} \vec{x}'_2 \right)^1 & \left(E^T \frac{\partial E}{\partial \psi} \vec{x}'_2 \right)^1 \\ \left(E^T \frac{\partial E}{\partial \phi} \vec{x}'_2 \right)^2 & \left(E^T \frac{\partial E}{\partial \theta} \vec{x}'_2 \right)^2 & \left(E^T \frac{\partial E}{\partial \psi} \vec{x}'_2 \right)^2 \\ \left(E^T \frac{\partial E}{\partial \phi} \vec{x}'_3 \right)^2 & \left(E^T \frac{\partial E}{\partial \theta} \vec{x}'_3 \right)^2 & \left(E^T \frac{\partial E}{\partial \psi} \vec{x}'_3 \right)^2 \end{pmatrix}, \quad (4.12)$$

the superindices standing for vector components, and

$$J_2^\mathcal{I} = \begin{pmatrix} \dots & \frac{\partial x_2'^3}{\partial q^b} & \dots \\ \dots & \frac{\partial x_3'^1}{\partial q^b} & \dots \\ \dots & \frac{\partial x_3'^3}{\partial q^b} & \dots \\ \dots & \frac{\partial \vec{x}'_4}{\partial q^b} & \dots \\ \dots & \vdots & \dots \\ \dots & \frac{\partial \vec{x}'_\alpha}{\partial q^b} & \dots \\ \dots & \vdots & \dots \end{pmatrix}. \quad (4.13)$$

The concrete form of the submatrix \mathcal{X} in eq. (4.11) is irrelevant for our purposes, since

$$\det J = \det J_2 = \det J_2^\mathcal{E} \det J_2^\mathcal{I}. \quad (4.14)$$

An explicit computation of $J_2^\mathcal{E}$ yields

$$J_2^\mathcal{E} = \begin{pmatrix} x_2'^3 \sin \theta \sin \psi & -x_2'^3 \cos \psi & 0 \\ x_2'^3 \sin \theta \cos \psi & x_2'^3 \sin \psi & 0 \\ -x_3'^1 \cos \theta + x_3'^3 \sin \theta \cos \psi & x_3'^3 \sin \psi & x_3'^1 \end{pmatrix}, \quad (4.15)$$

with determinant $\det J_2^\mathcal{E} = \sin \theta x_3'^1 (x_2'^3)^2$.

Using eqs. (4.14) and (4.6), we finally obtain

$$\det G(q^A, q^a) = \sin^2\theta (x_3'^1(q^a))^2 (x_2'^3(q^a))^4 \det J_2^T(q^a) \left(\prod_{\alpha=1}^n m_\alpha^3 \right), \quad (4.16)$$

where the factorization has been achieved, since the only factor that depends on the external coordinates is $\sin^2\theta$.

5 Conclusions

In this work, we have calculated explicit expressions in which the determinant of the mass-metric tensor G (eqs. (4.16) and (4.4)) and the determinant of the reduced mass-metric tensor g (eq. (3.19)), occurring in Classical Statistical Mechanics in the coordinate space, are written as a product of two functions; one depending only on the external coordinates that describe the overall translation and rotation of the system, and the other only on the internal coordinates. This has been done for any molecule, general internal coordinates and arbitrary constraints, extending the work in refs. 30 and 47.

This factorization allows to integrate out the external coordinates and perform Monte Carlo simulations in the internal conformational space, gaining insight of the problem, simplicity in the description of the system and, for small molecules, some computational effort. Also, our results indicate that, in general, the Fixman's compensating potential [31–34], customarily used to reproduce the stiff equilibrium distribution using rigid molecular dynamics simulations, does not depend on the external variables.

In appendix A, we give a general mathematical argument showing that the factorization is a consequence of the symmetries of the metric tensors involved and, in appendix B, the determinant of the mass-metric tensor G is computed explicitly in the SASMIC [24] set of curvilinear coordinates for general branched molecules (see eq. (A.10)) showing that the classical formula for serial polymers [30] is actually valid for any macromolecule.

All the expressions derived in the present work are directly applicable to real cases, as has been checked in ref. 31.

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Appendix A:

General mathematical argument

Let \mathcal{M} be a finite dimensional differentiable manifold equipped with a riemannian metric tensor. Take local coordinates q^μ on \mathcal{M} and denote by $G_{\mu\nu}(q)$ the components of the metric tensor in these coordinates.

The transformation

$$q'^\mu = q^\mu + \epsilon \xi^\mu(q) + O(\epsilon^2) \quad (A.1)$$

is said an *isometry* and $\xi^\mu(x)$ is said a *Killing vector field* if

$$G_{\mu\nu}(q'(q)) = J_{\mu}^{\rho}(q'(q)) G_{\rho\sigma}(q) J_{\nu}^{\sigma}(q'(q)) , \quad (\text{A.2})$$

where

$$J_{\nu}^{\mu}(q'(q)) := \left(\frac{\partial q^{\mu}}{\partial q'^{\nu}} \right) (q'(q)) . \quad (\text{A.3})$$

Expanding eq. (A.2) up to first order in ϵ and noticing that $\det(J_{\nu}^{\mu}) = 1 - \epsilon \partial_{\mu} \xi^{\mu}(q)$, we obtain the following differential equation for $\mathcal{G} := \det(G_{\mu\nu})$:

$$\xi^{\mu}(q) \partial_{\mu} \mathcal{G}(q) = -2(\partial_{\mu} \xi^{\mu}(q)) \mathcal{G}(q) . \quad (\text{A.4})$$

Let us apply this machinery to the case considered in this work. For concreteness, we shall derive the factorization of the external coordinates in the unconstrained case and shall argue that this still holds in the constrained one.

Simultaneous translations and rotations of all the particles² are isometries of the mass-matrix tensor in eq. (3.3). The important point for us is that, in the coordinates q^{μ} introduced in sec. 2, these transformations change the external coordinates $(X, Y, Z, \phi, \theta, \psi)$ and leave the internal coordinates q^a untouched (see eq. (2.2)).

A global translation is given in Cartesian coordinates by $x_{\alpha}^p \mapsto x_{\alpha}^p + \epsilon$. In the coordinates q^{μ} , it takes $(X, Y, Z) \mapsto (X, Y, Z) + \epsilon(1, 1, 1)$ and does not affect the remaining coordinates. With the above notation, $\xi^{\mu} = 1$, $\mu = 1, 2, 3$ and $\xi^{\mu} = 0$, $\forall \mu > 3$. Hence, eq. (A.4) implies that

$$\partial_X \mathcal{G} = \partial_Y \mathcal{G} = \partial_Z \mathcal{G} = 0 , \quad (\text{A.5})$$

i.e., the determinant of the mass-metric tensor does not depend on the coordinates X, Y, Z .

A global rotation in the coordinates q^{μ} rotates (X, Y, Z) and changes the Euler angles (in a complicated way which will not be important for our purposes) but does not affect the internal coordinates. Hence, $\xi^{\mu} = 0, \forall \mu > 6$. In addition, the matrix J_{ν}^{μ} does not depend on X, Y, Z because the rotation acts linearly on them. Let us abbreviate $\alpha \equiv \alpha^p \equiv (\phi, \theta, \psi)$, $p = 1, \dots, 3$. Recalling that \mathcal{G} does not depend on X, Y, Z , the differential equation (A.4) reads

$$\xi^p(\alpha) \partial_p \mathcal{G}(\alpha, q^a) = -2(\partial_p \xi^p(\alpha)) \mathcal{G}(\alpha, q^a) . \quad (\text{A.6})$$

The group of rotations in \mathbb{R}^3 has three linearly independent Killing vector fields which are complete in the sense that one can join two arbitrary points (ϕ, θ, ψ) and (ϕ', θ', ψ') by moving along integral curves of the Killing vector fields. This guarantees that the solution of eq. (A.6) is of the form

$$\mathcal{G}(\alpha, q^a) = \mathcal{G}_1(\alpha) \mathcal{G}_2(q^a) \quad (\text{A.7})$$

and we have the desired result.

To derive the factorization of the external coordinates in the constrained case, simply notice that the constraints in this work do not involve the external coordinates. Therefore, global translations and rotations are still isometries of the reduced mass-metric tensor and the result follows.

²Notice that the isometry group of the mass-metric tensor is much bigger, since translations and rotations acting independently on each particle are also isometry transformations.

Appendix B:

Determinant of G in particular coordinates

The SASMIC scheme, introduced in ref. 24, is a set of rules to define particular Z-matrix coordinates [59, 60] of general branched molecules, with convenient properties of modularity and approximate separability of soft and hard modes.

According to the rules, to each atom α , one uniquely assigns three atoms $\beta(\alpha)$, $\gamma(\alpha)$ and $\delta(\alpha)$ in such a way that the three Z-matrix internal coordinates that position atom α are

$$\begin{aligned} r_\alpha &:= (\alpha, \beta(\alpha)) \\ \theta_\alpha &:= (\alpha, \beta(\alpha), \gamma(\alpha)) \\ \phi_\alpha &:= (\alpha, \beta(\alpha), \gamma(\alpha), \delta(\alpha)) , \end{aligned} \quad (\text{A.1})$$

being r_α a bond length, θ_α a bond angle and ϕ_α a dihedral angle.

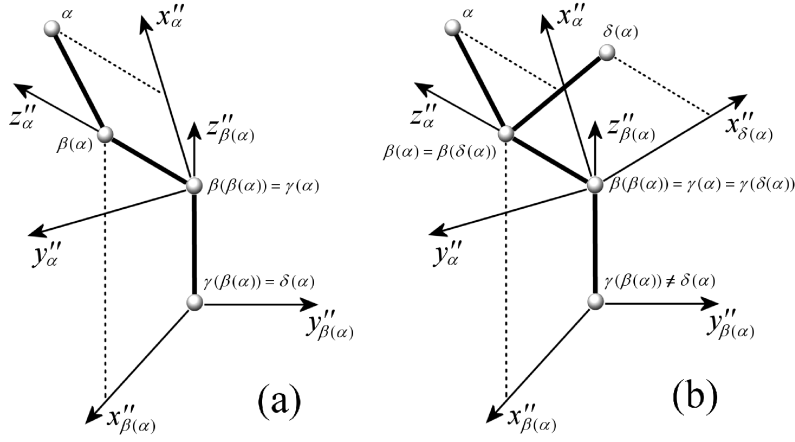


Figure 2: Local reference frames associated to atoms α and $\beta(\alpha)$ (see text) in the cases that (a) ϕ_α is a principal dihedral or (b) ϕ_α is a phase dihedral.

The procedure that will be followed in order to express the position \vec{x}'_α of atom α in the primed reference frame in fig. 1 as a function of the SASMIC internal coordinates starts by expressing the vector that goes from $\beta(\alpha)$ to α in a set of axes $(x''_\alpha, y''_\alpha, z''_\alpha)$ associated to α . This local reference frame is defined such that the z''_α -axis lies along the directional bond $(\gamma(\alpha), \beta(\alpha))$ and the x''_α -axis lies along the projection of $(\beta(\alpha), \alpha)$ onto the plane orthogonal to $(\gamma(\alpha), \beta(\alpha))$ (see fig. 2).

In these axes, the components of the vector $(\beta(\alpha), \alpha)$ are

$$\vec{x}''_\alpha{}^T := (r_\alpha \sin \theta_\alpha, 0, -r_\alpha \cos \theta_\alpha) . \quad (\text{A.2})$$

Now, if the atom $\delta(\alpha)$ that is used to define ϕ_α is bonded to atom $\gamma(\alpha)$ (fig. 2a), ϕ_α is called a *principal dihedral* [24] and we have that

$$\underbrace{\begin{pmatrix} -\cos \theta_{\beta(\alpha)} & 0 & \sin \theta_{\beta(\alpha)} \\ 0 & 1 & 0 \\ -\sin \theta_{\beta(\alpha)} & 0 & -\cos \theta_{\beta(\alpha)} \end{pmatrix}}_{\Theta(\theta_{\beta(\alpha)})} \underbrace{\begin{pmatrix} \cos \phi_\alpha & -\sin \phi_\alpha & 0 \\ \sin \phi_\alpha & \cos \phi_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\Phi(\phi_\alpha)} \vec{x}''_\alpha \quad (\text{A.3})$$

are the components of the vector $(\beta(\alpha), \alpha)$ in the local reference frame $(x''_{\beta(\alpha)}, y''_{\beta(\alpha)}, z''_{\beta(\alpha)})$ associated to atom $\beta(\alpha)$.

On the other hand, if we are at a branching point and the atom $\delta(\alpha)$ that is used to define ϕ_α is bonded to atom $\beta(\alpha)$ (fig. 2b), ϕ_α is called a *phase dihedral* [24] and we have to change first to the local reference frame associated to $\delta(\alpha)$. In this case, the components of the vector $(\beta(\alpha), \alpha)$ in the local reference frame $(x''_{\beta(\alpha)}, y''_{\beta(\alpha)}, z''_{\beta(\alpha)})$ are

$$\Theta(\theta_{\beta(\alpha)})\Phi(\phi_{\delta(\alpha)})\Phi(\phi_\alpha)\vec{x}''_\alpha. \quad (\text{A.4})$$

If we iterate the procedure, by changing the axes to the ones associated to the atom $\beta(\beta(\alpha))$, i.e., the β atom that correspond to $\beta(\alpha)$ according to the SASMIC scheme, and so on, we will eventually arrive to the set of axis (x''_3, y''_3, z''_3) (since, in the SASMIC scheme [24], we have that $\beta(\alpha) < \alpha$). Note however that, according to the definition of the local reference frame given in the preceding paragraphs, the one associated to atom 3 is *exactly* the primed reference frame in fig. 1.

Hence, let us define, for each atom α , a matrix R_α as the product of the matrices obtained using eqs. (A.3) and (A.4) and successively applying the function $\beta(\alpha)$. Then, R_α takes the vector $(\beta(\alpha), \alpha)$ in eq. (A.2) to the primed reference frame.

Let the superindex on β denote composition of functions, let us define $\beta^0(\alpha) := \alpha$ and \mathcal{N}_α as the integer such that $\beta^{\mathcal{N}_\alpha+1}(\alpha) = 3$. Adding all the vectors corresponding to $(\beta^{p+1}(\alpha), \beta^p(\alpha))$ in the primed reference frame, with $p = 0, \dots, \mathcal{N}_\alpha$, to \vec{x}'_3 yields the position of atom α in the primed reference frame as a function of the internal coordinates:

$$\vec{x}'_\alpha = \vec{x}'_3 + \sum_{p=\mathcal{N}_\alpha}^0 R_{\beta^p(\alpha)}\vec{x}''_{\beta^p(\alpha)}. \quad (\text{A.5})$$

Now, ordering the internal coordinates as $(r_2, r_3, \theta_3, r_4, \theta_4, \phi_4, \dots, r_n, \theta_n, \phi_n)$ and using the already mentioned fact that $\beta(\alpha) < \alpha$, and also that $\delta(\alpha) < \alpha$, we have that the matrix J_2^T in eq. (4.13) is

$$J_2^T = \begin{pmatrix} A_0 & & 0 \\ & A_4 & \\ & & \ddots \\ \mathcal{X} & & & A_n \end{pmatrix} \quad \text{and} \quad \det J_2^T = \det A_0 \prod_{\alpha=4}^n \det A_\alpha. \quad (\text{A.6})$$

Using that

$$\vec{x}'_2 = \begin{pmatrix} 0 \\ 0 \\ r_2 \end{pmatrix} \quad \text{and} \quad \vec{x}'_3 = \begin{pmatrix} r_3 \sin \theta_3 \\ 0 \\ r_2 - r_3 \cos \theta_3 \end{pmatrix}, \quad (\text{A.7})$$

we have

$$A_0 := \begin{pmatrix} \frac{\partial x_2'^3}{\partial r_2} & \frac{\partial x_2'^3}{\partial r_3} & \frac{\partial x_2'^3}{\partial \theta_3} \\ \frac{\partial x_3'^1}{\partial r_2} & \frac{\partial x_3'^1}{\partial r_3} & \frac{\partial x_3'^1}{\partial \theta_3} \\ \frac{\partial x_3'^3}{\partial r_2} & \frac{\partial x_3'^3}{\partial r_3} & \frac{\partial x_3'^3}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \theta_3 & r_3 \cos \theta_3 \\ 1 & -\cos \theta_3 & r_3 \sin \theta_3 \end{pmatrix}. \quad (\text{A.8})$$

Now, we note that the matrix $\Phi(\phi_\alpha)$ occurs always at the right-most place in R_α and that the derivatives in the blocks A_α , with $\alpha > 4$, kill all the terms in eq. (A.5) except for the one corresponding to $p = 0$. Hence, if we define $R_\alpha =: R'_\alpha \Phi(\phi_\alpha)$, the block A_α may be expressed as follows:

$$A_\alpha := R'_\alpha \begin{pmatrix} \frac{\partial \Phi(\phi_\alpha) \vec{x}''_\alpha}{\partial r_\alpha} & \frac{\partial \Phi(\phi_\alpha) \vec{x}''_\alpha}{\partial \theta_\alpha} & \frac{\partial \Phi(\phi_\alpha) \vec{x}''_\alpha}{\partial \phi_\alpha} \end{pmatrix} = \begin{pmatrix} \sin \theta_\alpha \cos \phi_\alpha & r_\alpha \cos \theta_\alpha \cos \phi_\alpha & -r_\alpha \sin \theta_\alpha \sin \phi_\alpha \\ \sin \theta_\alpha \sin \phi_\alpha & r_\alpha \cos \theta_\alpha \sin \phi_\alpha & r_\alpha \sin \theta_\alpha \cos \phi_\alpha \\ -\cos \theta_\alpha & r_\alpha \sin \theta_\alpha & 0 \end{pmatrix}. \quad (\text{A.9})$$

Finally, using eq. (A.6), noting that $\det A_0 = r_3$ and $\det A_\alpha = -r_\alpha^2 \sin \theta_\alpha$, and calculating the remaining terms of eq. (4.16) with eq. (A.7), we obtain the desired result:

$$\det^{\frac{1}{2}} G(q^A, q^a) = \left(\prod_{\alpha=1}^n m_\alpha^{3/2} \right) |\sin \theta| \left(\prod_{\alpha=2}^n r_\alpha^2 \right) \left(\prod_{\alpha=3}^n |\sin \theta_\alpha| \right). \quad (\text{A.10})$$

It is worth remarking at this point that the previous expression does not explicitly depend on the dihedral angles ϕ_α and that it is the same result as the one found in ref. 30 for serial polymers.

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