HERMITE EXPANSIONS OF ELEMENTS OF GENERALIZED GELFAND-SHILOV SPACE QUASIANALYTIC AND NON QUASIANALYTIC CASE

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ABSTRACT. We characterize the elements of generalized Gelfand Shilov spaces in terms of the coefficients of their Fourier-Hermite expansion. The technique we use can be applied both in quasianalytic and nonquasianalytic case. The characterizations imply the kernel theorems for the dual spaces. The cases when the test space is quasianalytic are important in quantum field theory with a fundamental length, since the properties of the space of Fourier hyper functions, which is isomorphic with Gelfand-Shilov space S_1^1 are well adapted for the use in the theory, see papers of E.Bruning and S.Nagamachi.

1. INTRODUCTION

In order to study classes of functionals invariant under the Fourier transform, but larger then the classes of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, I.M. Gelfand and G.E Shilov ([\[5\]](#page-18-0)) introduced spaces $\mathcal{S}_{\alpha}^{\alpha}(\mathbb{R}^d)$, $\alpha \geq 1/2$. Their topological duals have been successfully used in differential operators theory and in spectral analysis. In the special case, when the test spaces are non-quasianalytic, i.e. when $\alpha > 1$, Gelfand-Shilov spaces were also successfully used in the framework of time-frequency analysis (see [\[12\]](#page-18-1) and references there). But the cases when the test space is quasianalytic (i.e. when $\alpha \in [1/2, 1]$ are also very important for applications, see for example [\[4\]](#page-18-2) and [\[9\]](#page-18-3), where it was conjectured that the properties of the space of Fourier hyper functions, which is isomorphic with S_1^1 are well adapted for the use in quantum field theory with a fundamental length.

In the paper we study the generalized Gelfand-Shilov $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and the generalized Pilipović $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ spaces and their duals, which generalize all nontrivial Gelfand-Shilov $\mathcal{S}_{\alpha}^{\alpha}$ and Pilipović spaces $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$ ([\[10\]](#page-18-4)) in quasianalytic and nonquasianalytic case in an uniform way. These spaces are subclasses of Denjoy-Carleman classes $C^{\{M_p\}}(\mathbb{R}^d)$ and $C^{(M_p)}(\mathbb{R}^d)$ which are invariant under Fourier transform, closed under differentiation and multiplication by polynomials, and equipped with appropriate topologies.

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The duals $S^{\{M_p\}'}(\mathbb{R}^d)$ and $S^{(M_p)'}(\mathbb{R}^d)$, are good spaces for harmonic analysis, they are invariant under Fourier transform and have the space of tempered distributions as a proper subspace.

The aim of the paper is to characterize the dual spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ in terms of the Hermite coefficients of their elements. The elegant proofs of kernel theorems for the spaces are a consequence of the characterizations. The kernel theorem imply that the representation of the Heisenberg group and the Weyl transform can be extended to the spaces of tempered ultradistributions. The simple nature of the proofs depends on extensive use of the Hermite expansion of elements of the spaces.

The examples of the spaces $S^{\{M_p\}}(\mathbb{R}^d)$ and $S^{(M_p)}(\mathbb{R}^d)$ are:

- for $M_p = p^{\alpha p}$, the space $S^{\{M_p\}}(\mathbb{R}^d)$ is the Gelfand-Shilov space S^{α}_{α} and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the Pilipović space \sum_{α}^{α} ;
- for $M_p = p^p$ then the space $S^{\{M_p\}}(\mathbb{R}^d)$ is isomorphic with the Sato space \mathcal{F} , the test space for Fourier hyperfunctions \mathcal{F}' , and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the Silva space G , the test space for extended Fourier hyperfunctions \mathcal{G}' ;
- Braun-Meise-Taylor space $\mathcal{S}_{\{\omega\}}, \omega \in \mathcal{W}$, introduced in [\[1\]](#page-18-5) and studied in the series of papers by the same authors, is the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, where

$$
M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.
$$

The sequence satisfies the conditions (M.1), (M.2) and (M.3)', and it is in general different from a Gevrey sequence.

• Beurling-Björk space $\mathcal{S}_{\omega}, \omega \in \mathcal{M}_c$, introduced in [\[3\]](#page-18-6), is equal to the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where

$$
M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.
$$

The sequence satisfies the conditions $(M.1)$ and $(M.3)$ ['], and it is in general different from a Gevrey sequence. If we assume additionally that $\omega(\rho) \ge C(\log \rho)^2$ for some $C > 0$, then $(M.2)$ is satisfied.

• In [\[8\]](#page-18-7) Korevaar developed a very general theory of Fourier transforms, based on a set of original and well motivated ideas. In order to obtain a formal class of objects which contain functions of exponential growth and which is closed under Fourier transform he introduced objects called pansions of exponential growth. From characterization theorem [\[8,](#page-18-7) Theorem 92.1] and our results it follows that exponential pansions are exactly tempered ultradistributions of Roumieu-Komatsu type, for $M_p = p^{p/2}$.

In the paper the sequence $\{M_p\}_{p \in \mathbb{N}_0}$, which generates the Denjoy-Carleman classes $C^{\{M_p\}}(\mathbb{R}^d)$ and $C^{(M_p)}(\mathbb{R}^d)$ is a sequence of positive numbers. We suppose that it satisfies the first two standard conditions in ultradistributional

theory: the conditions $(M.1)$ - logarithmic convexity and $(M.2)$ - separativity condition. We do not suppose nonquasianaliticity of Denjoy-Carleman classes $C^{\{M_p\}}(\mathbb{R}^d)$ and $C^{(M_p)}(\mathbb{R}^d)$ of functions, which is the standard nontriviality condition in the theory of ultradistributions (the condition (M.3)' in $[7]$). Instead, we suppose weaker condition $(M.3)$ " (resp. $(M.3)$ "'), which is minimal nontriviality condition appropriate for the spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$, (resp. $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$). Introduction of conditions $(M.3)$ ", (resp. $(M.3)$ "') gives us a possibility to treat the quasianalytic and nonquasinanalytic cases in the unified way. In nonquasianalytic case the dual space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is the space of tempered ultradistributions and in quasianalytic case elements of $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ are hyperfunctions.

An example of a class of sequences which satisfy the above conditions is:

(1.1)
$$
M_p = p^{sp} (\log p)^{tp}, \quad p \in \mathbb{N}, \quad s \ge 1/2, \ t \ge 0,
$$

and (only) in Beurling-Komatsu case we assume additionally $s + t > 1/2$.

If the nonquasianalytic condition $(M.3)$ ' is also satisfied, the spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ are the proper subspaces of Roumieu-Komatsu and of Beurling-Komatsu ultradistributions (see [\[7\]](#page-18-8)). If however, the condition (M.3)' is not satisfied, these spaces of ultradistributions are trivial, nevertheless the spaces which are studied in this paper are not.

In Section 2. we prove the basic identification of the generalized Gelfand-Shilov space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and its dual space, with the sequence spaces of the Fourier-Hermite coefficients of their elements. First we prove that the test space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, can be identified with the space of sequences of ultrafast falloff, i.e. (in one dimensional case) the space of sequences of complex numbers $\{a_n\}_{n\in\mathbb{N}_0}$ which satisfy that for some $\theta > 0$

$$
\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta\sqrt{n})] < \infty.
$$

Here, $M(\cdot)$ is the associated function for the sequence $\{M_p\}_{p\in\mathbb{N}_0}$ defined by

(1.2)
$$
M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \quad \rho > 0.
$$

In the special case [\(1.1\)](#page-2-0), one have $M(\rho) = \rho^{\frac{1}{s}} (\log \rho)^{-\frac{t}{s}}, \rho \gg 0.$

Next we prove that the dual space $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ can be identified with the space of sequences of ultrafast growth, i.e. (in one dimensional case) the space of sequences $\{b_n\}_{n\in\mathbb{N}_0}$ which satisfy that for every $\theta > 0$

$$
\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{n})] < \infty.
$$

There is an analogy between generalized Gelfand-Shilov and generalized Pilipovic spaces. One can modify the results obtained for one type of spaces to another, but there are differences, about which one should take care. Therefore, in Section 3 we obtain sequential characterization of generalized

Pilipović spaces $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and state the kernel theorem for the spaces of tempered ultradistributions of Beurling-Komatsu type.

As an application of the sequential characterizations of generalized Gelfand-Shilov spaces, we state in Section 4 the kernel theorem for tempered ultradistributions of Roumieu - Komatsu type.

In the last section we give the proofs of the two essential lemmas. The first one gives appropriate estimation for the growth of the derivatives of Hermite functions. In the paper we need sharper estimations for the derivatives of Hermite functions then the estimations usually given in the literature (see for example [\[16,](#page-18-9) p 122]). In the second lemma we estimate action of an ultradifferential operator, which is generated by creation and annihilation operators.

1.1. Notations and basic notions. Throughout the paper by C we denote a positive constant, not necessarily the same at each occurrence. Let ${M_p}_{p \in \mathbb{N}_0}$ be a sequence of positive numbers, where $M_0 = 1$.

Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ (see [\[13\]](#page-18-10)) is a class of smooth functions φ such that there exist $m > 0$ and $C > 0$ so that

(1.3)
$$
||\varphi^{(\alpha)}||_{\infty} \leq Cm^{|\alpha|}M_{|\alpha|}, \quad |\alpha| \in \mathbb{N}^d,
$$

where we use multi-index notation:

$$
\varphi^{(\alpha)}(x) = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \cdots (\partial/\partial x_d)^{\alpha_d} \varphi(x).
$$

and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$. The class of functions equipped with a natural topology is the space of ultradifferentiable functions of Roumie-Komatsu type $\mathcal{E}^{\{M_p\}}(\mathbb{R}^d)$ (for the definition see [\[7\]](#page-18-8)). In the special case when $\{M_p\}_{p \in \mathbb{N}_0}$ is a Gevrey sequence $\{p^{sp}\}_{p \in \mathbb{N}_0}$ the space is the Gevrey space $\mathcal{G}^{\{s\}}(\mathbb{R}^d)$.

In the paper we define the generalized Gelfand-Shilov space as subclasses of the Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ invariant under Fourier transform, closed under the differentiation and multiplication by a polynomial, and equip them with appropriate topologies.

In the paper we assume that the sequence $\{M_p\}_{p\in\mathbb{N}_0}$ satisfy

- (M.1) $M_p^2 \le M_{p-1}M_{p+1}, \quad p = 1, 2, \dots$ (logarithmic convexity)
- (M.2) There exist constants $A, H > 0$ such that $M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \ldots$ (separativity condition or stability under ultradifferential operators)
- $(M.3)$ " There exist constants $C, L > 0$ such that $p^{\frac{p}{2}} \leq C L^p M_p, \quad p = 0, 1, ...$ (non triviality condition for the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$)

In Section 4, where we discus generalized Pilipović spaces, instead of $(M.3)$ " we assume:

 $(M.3)$ "' For every $L > 0$, there exists $C > 0$ such that $p^{\frac{p}{2}} \leq CL^p M_p, \quad p = 1, 2, \dots$ (non triviality condition for the spaces $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$).

To be able to discuss our results in the context of Komatsu's ultradistributions, let us state condition :

 $(M.3)^\circ \sum_{p=1}^{\infty}$ M_{p-1} $\frac{q_{p-1}}{M_p}<\infty.$ (non-quasi-analyticity)

The condition (M.1) is of technical nature, which simplify the work and involve no loss of generality. This is the well known fact for the Denjoy-Carleman classes of functions, (see for example [\[13\]](#page-18-10)).

The condition (M.2) is standard in the ultradistribution theory. It implies that the class $C^{\{M_p\}}(\mathbb{R}^d)$ is closed under the (ultra)differentiation (see [\[7\]](#page-18-8)), and is important in characterization of Denjoy-Carleman classes in multidimensional case.

The non-triviality conditions $(M.3)$ " and $(M.3)$ " are weaker then the condition $(M.3)$. Under the conditions $(M.3)$ " and $(M.3)$ " all Hermite functions are elements of the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ respectively. The smallest nontrivial Gelfand-Shilov space is $S_{1/2}^{1/2}$ $_{1/2}^{1/2}(\mathbb{R}^d)$. Condition (M.3)" essentially means that the space $S_{1/2}^{1/2}$ $_{1/2}^{1/2}(\mathbb{R}^d)$ is a subset of $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$. The smallest nontrivial Pilipović space does not exist. Note, $\sum_{1/2}^{1/2} = \{0\}$, but the space $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$, $\alpha > 1/2$, is nontrivial. Moreover, every nontrivial Pilipović space $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$, contains as a subspace one generalized Pilipović space, for example, the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where $M_p = p^{p/2} (\log p)^{pt}$, $t > 0$.

The condition (M.3)' is necessary and sufficient condition that the classe $C^{\{M_p\}}(\mathbb{R}^d)$ has a nontrivial subclass of functions with compact support, i.e. that $C^{\{M_p\}}(\mathbb{R}^d)$ is non-quasianalytic class of functions.

For example, the sequence (1.1) satisfies conditions $(M.1)$, $(M.2)$, $(M.3)$ ", and if $t > 0$ also the condition $(M.3)$ " but not $(M.3)$; and for $s > 1$ it satisfies the stronger condition (M.3)'.

2. Generalized Gelfand-Shilov spaces

2.1. **Basic spaces.** We define the set $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ as a subclass of the Denjoy-Carleman class $C^{\{M_p\}}(\mathbb{R}^d)$ which is invariant under Fourier transform, closed under the differentiation and multiplication by a polynomial. This imply that it is a subset of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ od rapidly decreasing functions. and therefore of every $L^q(\mathbb{R}^d)$, $q \in [1,\infty]$. The same set can be characterized in one of the following equivalent ways:

1. The set $\mathcal{S}^{\{M_p\}}_{\cdot}(\mathbb{R}^d)$ is the set of all smooth functions φ such that there exist $C > 0$ and $m > 0$ such that

$$
\|\exp[M(m\,x)]\varphi\|_2 < C \quad \text{and} \quad \|\exp[M(m\,x)]\mathcal{F}\varphi\|_2 < C,
$$

where $|| \cdot ||_2$ is the usual norm in $L(\mathbb{R}^d)$, $\mathcal F$ is the Fourier transform and the function $M(\cdot)$ is defined by [\(1.2\)](#page-2-1).

2. The set $\mathcal{S}^{\{M_p\}}$ is the set of all smooth functions φ on \mathbb{R}^d , such that for some $C > 0$ and $m > 0$

$$
(2.1) \qquad || (1+x^2)^{\beta/2} \varphi^{(\alpha)} ||_{\infty} \le C m^{|\alpha|+|\beta|} M_{|\alpha|} M_{|\beta|}, \text{ for every } \alpha, \beta \in \mathbb{N}_0^d.
$$

The topology of the generalized Gelfand-Shilov space is the inductive limit topology of Banach spaces $\mathcal{S}^{M_p,m}$, $m > 0$, where by $\mathcal{S}^{M_p,m}$, we denote the space of smooth functions φ on \mathbb{R}^d , such that for some $C > 0$ and $m > 0$

(2.2)
$$
||\varphi||_{\mathcal{S}^{M_p},m} = \sup_{\alpha,\beta \in \mathbb{N}_0^d} \frac{m^{|\alpha|+|\beta|}}{M_{|\alpha|}M_{|\beta|}} ||(1+x^2)^{\beta/2} \varphi^{(\alpha)}(x)||_{L^{\infty}} < \infty,
$$

equipped with the norm $|| \cdot ||_{\mathcal{S}^{M_p},m}$. So, $\mathcal{S}^{\{M_p\}} = ind\,lim_{m \to 0} \mathcal{S}^{M_p,m}$

It is a Frechet space. We will denote by $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ the strong dual of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and call it the space of the **tempered ultradistributions** of Roumieu-Komatsu type.

Fourier transform is defined on $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ by

$$
\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{ix\xi} f(x) dx, \quad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d),
$$

and on $\mathcal{S}^{\{M_p\}'}$ by

$$
\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle, \ \ f \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^d), \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)
$$

The space is a good space for harmonic analysis since the Fourier transform is an isomorphism of $S^{\{M_p\}}(\mathbb{R}^d)$ onto itself, and the space of tempered distributions \mathcal{S}' is its proper subspace.

2.2. Hermite functions. We denote by

$$
\mathcal{H}_n(x) = (-1)^n \pi^{-1/4} 2^{-n/2} (n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2} \right), \quad n \in \mathbb{N},
$$

the Hermite functions (the wave functions of a harmonic oscillator), where $\mathcal{H}_{-k} = 0$ for $k = 1, 2, 3...$ The functions arise naturally as eigenfunctions of harmonic oscillator Hamiltonian, and so play a vital role in quantum physics, but they are also eigenfunctions of the Fourier transform. This fact will be used often in the paper.

In the paper we will use the properties of the creation and the annihilation operators:

:

$$
L^{+} = \frac{1}{\sqrt{2}}(x - \frac{d}{dx}), \quad L^{-} = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})
$$

(1.1) $L^{-}L^{+} - L^{+}L^{-} = 1$,
(1.2) $L^{-}\mathcal{H}_{n} = \sqrt{n}\,\mathcal{H}_{n-1}, \quad L^{+}\mathcal{H}_{n} = \sqrt{n+1}\,\mathcal{H}_{n+1},$

(1.3) $L^+L^-{\mathcal H}_n=n{\mathcal H}_n,$

 (1.1)

the fact that the sequence $\{\mathcal{H}_n\}_{n\in\mathbb{N}_0}$ is an orthonormal system in $L^2(\mathbb{R})$ and

$$
\mathcal{F}[\mathcal{H}_n] = \sqrt{2\pi} i^n \mathcal{H}_n.
$$

The Hermite functions in multidimensional case are defined simply by taking the tensor product of the one dimensional Hermite functions:

$$
\mathcal{H}_n(x)=\mathcal{H}_{n_1}(x_1)\mathcal{H}_{n_2}(x_2)\cdots\mathcal{H}_{n_d}(x_d), \quad x=(x_1,x_2,...x_d)\in\mathbb{R}^d,
$$

where $n = (n_1, n_2, ..., n_d) \in \mathbb{N}^d$. The functions \mathcal{H}_{n_i} , $n \in \mathbb{N}_0^d$, are elements of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$. This an immediate consequence of the Lemma [2.1.](#page-6-0)

Let φ be a smooth function of fast falloff $(\varphi \in \mathcal{S}(\mathbb{R}^d))$. The numbers

$$
a_n(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \mathcal{H}_n(x) dx, \quad n \in \mathbb{N}_0^d
$$

will be called the **Fourier-Hermite coefficients** of φ . The sequence of the Fourier-Hermite coefficients ${a_n(\varphi)}_{n \in \mathbb{N}_0^d}$ of φ we call the **Hermite repre**sentation of φ .

We will extensively use the following estimations, which we prove in the last section.

Lemma 2.1. a) If conditions $(M.1)$, $(M.2)$ and $(M.3)$ " are satisfied, there exist $C > 0$ and $m_0 > 0$ such that for every $m \leq m_0$

(2.3)
$$
\frac{m^{\alpha+\beta}}{M_{\alpha}M_{\beta}}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \leq C e^{M(8mH\sqrt{n})}.
$$

b) If conditions (M.1), (M.2) and (M.3)"' are satisfied, for every $m > 0$ there exists $C > 0$ such that the estimate [\(2.3\)](#page-6-1) holds.

We will also need the following lemma:

Lemma 2.2. a) If $\varphi \in C^{\infty}$ and $N \in \mathbb{N}$ then

$$
(2.4) \quad (L^-L^+)^N \varphi(x) = 2^N (1+x^2 - \frac{d^2}{dx^2})^N \varphi(x) = \sum_{p=0}^{2N} \sum_{q=0}^{2N-p} c_{p,q}^{(N)} x^p \varphi^{(q)}(x),
$$

where $c_{p,q}^{(N)}$ are constants which satisfy inequality

(2.5)
$$
|c_{p,q}^{(N)}| \le 26^N (2N-q)^{N-\frac{p+q}{2}}.
$$

b) Moreover, if conditions (M.1), (M.2) and (M.3)" are satisfied for $p, q \in \mathbb{N}$, $p + q \leq 2N$, then it holds:

(2.6)
$$
|c_{p,q}^{(N)}| \leq 52^N \frac{M_N^2}{M_p M_q}.
$$

2.3. Hermite representation of generalized Gelfand-Shilov space. The fact that the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is isomorphic with sequence space s of sequences of fast falloff, has a lot of important consequences (see for example [\[14\]](#page-18-11), [\[15\]](#page-18-12) and [\[16\]](#page-18-9)). One of them is a simple proof of the kernel theorem for the space of tempered distributions [\[14\]](#page-18-11). An analogue of that property hold for generalized Gelfand-Shilov and generalized Pilipović spaces. In this section we will prove this fact.

In the paper by $\mathbf{s}_{M_p,\theta}$, $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d_+$, we denote the set of multisequences ${a_n}_{n \in \mathbb{N}_0^d}$ of complex numbers which satisfies that

$$
\|\{a_n\}\|_{\theta} = \left(\sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp\left[\sum_{k=1}^d M(\theta_k \sqrt{n_k})\right]\right)^{1/2} < \infty,
$$

equipped with the norm $\|\{a_n\}\|_{\theta}$.

The space $\mathbf{s}_{\{M_p\}}$ of sequences of ultrafast falloff is the inductive limit of the family of spaces $\{s_{M_p,\theta}, \theta \in \mathbb{R}^d_+\}$, and it is a nuclear space (see [\[15\]](#page-18-12)).

Theorem 2.3. The mapping which assigns to each element of $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ its Hermite representation is a topological isomorphism of the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and the space $s_{\{M_n\}}$ of sequences of ultrafast falloff.

Space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is nuclear, since the space $\mathbf{s}_{\{M_p\}}$ is nuclear.

We will prove Theorem [2.3](#page-7-0) in one dimensional case, the proof in multidimensional is an immediate consequence of it.

Proof. 1. Let $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$ then there exists $\mu > 0$ such that

$$
||\varphi||_{M_p,\mu} = \sup_{p,q} \frac{\mu^{p+q}}{M_p M_q} ||(1+x^2)^{p/2} \varphi^{(q)}(x)||_{\infty} < \infty.
$$

From the property (1.3) of the creation and annihilation operators it follows that

(2.7)
$$
a_n(\varphi) = \int \varphi(x) \mathcal{H}_n(x) dx = n^{-N} \int \varphi(x) (L^+ L^-)^N \mathcal{H}_n(x) dx =
$$

$$
= n^{-N} \int \left((L^- L^+)^N \varphi(x) \right) \mathcal{H}_n(x) dx =
$$

$$
= n^{-N} \int (1 + x^2) \left((L^- L^+)^N \varphi(x) \right) \mathcal{H}_n(x) \frac{dx}{1 + x^2}.
$$

From Lemma [2.2](#page-6-2) and condition (M.2) it follows that

$$
(1+x^2)|(L^-L^+)^N\varphi(x)| \le
$$

\n
$$
\leq 52^N M_N^2 \sum_{p=0}^{2N} \sum_{q=0}^{2N-p} \frac{(1+x^2)|x^p\varphi^{(q)}(x)|}{M_pM_q} \leq
$$

\n
$$
\leq C 52^N H^{2N} M_N^2 \sum_{p=0}^{2N} \sum_{q=0}^{2N-p} \frac{\mu^{p+q}(1+x^2)^{(p+2)/2}\varphi^{(q)}(x)}{M_{p+2}M_q} \mu^{-(p+q)} \leq
$$

$$
\leq C \, \theta^N M_N^2 ||\varphi||_{M_p,\mu},
$$

where $\theta = \sqrt{52}H \cdot 2 \cdot (1 + \mu)$.

Since $||\mathcal{H}_n||_{L^2} = 1$, from above it follows that for each $N \in \mathbb{N}_0$

$$
|a_n(\varphi)|^2 \le C \ n^{-2N} \theta^{2N} M_N^4 ||\varphi||_{M_p,\mu}^2,
$$

where $\theta = \sqrt{26}H \cdot 2 \cdot (1 + \mu)$. Therefore, for $N = \alpha + 2$ by (M.2) and (M.1) $|a_n(\varphi)|^2 \leq C n^{-2\alpha} n^{-2} \theta^{2\alpha} M_{\alpha}^4 H^{4\alpha} ||\varphi||^2_{M_p,\mu} \leq C n^{-2\alpha} n^{-2} (H^2 \theta)^{2\alpha} M_{2\alpha}^2 ||\varphi||^2_{M_p,\mu}$

which imply

$$
||\{a_n\}||_{\theta} = \left(\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(H^2\theta\sqrt{n})]\right)^{1/2} \le C ||\varphi||_{M_p,\mu} \le \infty,
$$

for $\theta = \sqrt{52}H \cdot 2 \cdot (1 + \mu)$.

2. Let for some $\theta > 0$ the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfy

$$
||\{a_n\}||_{\theta} = \left(\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta\sqrt{n})]\right)^{1/2} < \infty.
$$

It follows that the sequence is a sequence of fast falloff, so the sum $\sum_{n=0}^{\infty} a_n \mathcal{H}_n(x)$ converges to some φ in S. We will prove that the φ also belong to the space ${\cal S}^{\{M_p\}}(\mathbb{R}).$

Let m_0 and C be positive be the constants such that for every $m \leq m_0$ holds:

$$
(2.8) \qquad \frac{m^{\alpha+\beta}}{M_{\alpha}M_{\beta}}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \leq C \, \exp[M(8mH\sqrt{n})].
$$

existence of which is determined by Lemma [2.1.](#page-6-0) By using Cauchy - Schwartz inequality and Lemma [2.1](#page-6-0) we have:

$$
\frac{m^{\alpha+\beta}}{M_{\alpha}M_{\beta}}||(1+x^2)^{\beta/2} \left(\sum_{n=0}^{\infty} a_n \mathcal{H}_n\right)^{(\alpha)} ||_{\infty} \le
$$

\n
$$
\leq C \sum |a_n|_{n=0}^{\infty} \exp\left[M\left(8mH\sqrt{n}\right)\right] \le
$$

\n
$$
\leq C \left(\sum_{n=0}^{\infty} |a_n|^2 \exp\left[2M\left(\theta\sqrt{n}\right)\right]\right)^{1/2}.
$$

\n
$$
\cdot \left(\sum_{n=0}^{\infty} \exp[-2M(\theta\sqrt{n})] \exp\left[M\left(8mH\sqrt{n}\right)\right]\right)^{1/2}
$$

.

Since $\exp[-M(\theta\sqrt{n})] \leq C$, it follows that for $m < \theta/(8h)$

$$
||\varphi||_{M_p,m} = \sup_{\alpha,\beta} \frac{m^{\alpha+\beta}}{M_\alpha M_\beta} ||(1+x^2)^{\beta/2} \left(\sum_{n=0}^{\infty} a_n \mathcal{H}_n\right)^{(\alpha)} ||_{\infty} \le
$$

$$
\leq C \left(\sum_{n=0}^{\infty} |a_n|^2 \exp\left[2M\left(\theta\sqrt{n}\right)\right]\right)^{1/2} = ||\{a_n\}||_{\theta}.
$$

This concludes the proof of the second part of the theorem. QED

Let f be an element of the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$. The numbers

$$
a_n(f) = \langle f, h_n \rangle, \quad n \in \mathbb{N}_0^d.
$$

will be called the **Fourier-Hermite coefficients** of f, the sequence $\{a_n(f)\}_{n\in\mathbb{N}_0^d}$, the **Hermit representation** of f , and the formal series

$$
\sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{H}_n(x)
$$

will be called the Hermite series of f .

Let us now characterize the Hermite representation of $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$.

Theorem 2.4. 1. If $f \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ then for every $\theta = (\theta_1, \dots \theta_d) \in \mathbb{R}^d_+$ its Hermite representation $\{a_n\}_{n\in\mathbb{N}_0^d}$ satisfy

(2.9)
$$
|a_n(f)| \le \exp\left[\sum_{k=1}^d M(\theta_k \sqrt{n_k})\right], \quad n = (n_1, ... n_d)
$$

and f has the representation:

$$
\langle f, \varphi \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d),
$$

where the sequence ${a_n(\varphi)}_{n \in \mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$.

2. Conversely, if a sequence ${b_n}_{n\in\mathbb{N}_0^d}$ satisfy that for every $\theta=(\theta_1,...\theta_d)\in$ $\mathbb{R}^d_+,$

(2.10)
$$
|b_n| \le \exp\left[\sum_{k=1}^d M(\theta_k \sqrt{n_k})\right],
$$

it is the Hermite representation of a unique $f \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ and the **Parse**val equation holds:

$$
\langle f, \varphi \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d),
$$

where the sequence $\{a_n(\varphi)\}_{n\in \mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$.

Proof. For simplicity we will give the proof in one dimensional case. 1. Let $f \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$ and let $\theta > 0$. Then for every $\mu > 0$ there exists $C > 0$ such that

$$
|\langle f, \varphi \rangle| \leq C ||\varphi||_{M_p, \mu},
$$

for every $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R})$. From above and Lemma [2.1](#page-6-0) it follows that there exists $m_0 > 0$ and $C > 0$ such that for $m < \min(m_0, \theta/(8m))$

$$
|a_n(f)| = |\langle f, \mathcal{H} \rangle| \le C \sup_{\alpha, \beta} \frac{m^{\alpha+\beta}}{M_{\alpha}M_{\beta}} ||(1+x^2)^{\beta/2} \mathcal{H}_n^{(\alpha)}||_{\infty} \le
$$

$$
\le C \exp \left[M(8mH\sqrt{n}) \right] \le C \exp \left[M(\theta\sqrt{n}) \right].
$$

2. Let the sequence ${b_n}$ satisfy condition [\(2.10\)](#page-9-0) for every $\theta > 0$. We will prove that the series $\sum_{n=0}^{\infty} b_n \mathcal{H}_n$ converges in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ to an element of the space $S^{\{M_p\}'}(\mathbb{R})$ defined by

(2.11)
$$
f: \varphi \mapsto \sum_{n=0}^{\infty} b_n a_n(\varphi),
$$

where $\{a_n(\varphi)\}\$ is the Hermit representation of φ . From the Schwartz inequality it follows that for every $\theta > 0$

$$
\sum_{n=0}^{\infty} |b_n||a_n(\varphi)| \le
$$

$$
\leq \left(\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{n})]\right)^{1/2} \cdot \left(\sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{n})]\right)^{1/2} \le
$$

$$
\leq C\left(\sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{n})]\right)^{1/2} \leq C||\varphi||_{\theta},
$$

which implies that the mapping f defined by (2.11) is an element from $\mathcal{S}^{\{M_p\}'}(\mathbb{R}).$

The equation $f = \sum_{n=0}^{\infty} b_n \mathcal{H}_n$ holds in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$, since

$$
\langle f, \varphi \rangle = \lim_{k \to \infty} \sum_{n=0}^{k} b_n a_n(\varphi) = \lim_{k \to \infty} \sum_{n=0}^{k} b_n \langle \mathcal{H}_n, \varphi \rangle =
$$

$$
= \lim_{k \to \infty} \langle \sum_{n=0}^{k} b_n \mathcal{H}_n, \varphi \rangle,
$$

by virtue of the completeness of the space $S^{\{M_p\}}'(\mathbb{R})$ we have that $f =$ $\sum_{n=0}^{\infty} b_n \mathcal{H}_n$ in the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$. QED

3. Generalized Pilipovic space ´

The definition of Denjoy-Carleman class $C^{(M_p)}(\mathbb{R}^d)$ differs slightly from the standard one. It is a class functions φ such that for every $m > 0$ there exists $C > 0$ so that equation [\(1.3\)](#page-3-0) holds. The class of functions equipped with a natural topology is the space of ultradifferentiable functions of Beurling-Komatsu type $\mathcal{E}^{(M_p)}(\mathbb{R}^d)$ (see [\[7\]](#page-18-8)). In the special case, when ${M_p}_{p ∈ ℕ_0}$ is a Gevrey sequence ${p^{sp}}_{p ∈ ℕ_0}$, the space is the Gevrey space $\mathcal{G}^{(s)}(\mathbb{R}^d).$

We define the set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ as a subclass of the Denjoy-Carleman class $C^{(M_p)}(\mathbb{R}^d)$ which is invariant under Fourier transform, and closed under the differentiation and multiplication by a polynomial. Analogously as in Section 2., one can characterize the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ in one of the following equivalent ways:

1. The set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the set of all smooth functions φ such that for every $m > 0$ there exists $C > 0$ so that

$$
||\exp[M(mx)]\varphi||_2 < C \quad and \quad ||\exp[M(mx)]\mathcal{F}\varphi||_2 < C.
$$

2. The set $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is the set of all smooth functions φ on \mathbb{R}^d , such that for every $m > 0$ there exists $C > 0$ so that

$$
(3.1) \qquad || (1+x^2)^{\beta/2} \varphi^{(\alpha)} ||_{\infty} \le C m^{|\alpha|+|\beta|} M_{|\alpha|} M_{|\beta|}, \text{ for every } \alpha, \beta \in \mathbb{N}_0^d.
$$

The topology generalized Pilipović space is the projective limit topology of Banach spaces $\mathcal{S}^{M_p,m}$, $m > 0$, where by $\mathcal{S}^{M_p,m}$, is defined as in Section 2. Let us stress that every nontrivial Pilipovic space $\sum_{\alpha}^{\alpha}(\mathbb{R}^d)$, contains as a subspace one generalized Pilipović space, for example, the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where $M_p = p^{p/2} (\log p)^{pt}$.

We will denote by $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ the strong dual of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$. It contains space of tempered distributions as a proper subspace and the Fourier transform maps it into itself.

Analogously as in Section 2, one can prove following theorem:

Theorem 3.1. The mapping which assigns to each element of $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ its Hermite representation is a topological isomorphism of the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ and the space $\mathbf{s}_{(M_p)}$ of sequences of ultrafast falloff, where the space $\mathbf{s}_{(M_p)}$ is the space of sequences of ultrafast falloff and is the projective limit of the family of spaces $\{s_{M_p,\theta}, \theta \in \mathbb{R}^d_+\}$, which are defined in Section 2.

Since the space $\mathbf{s}_{(M_p)}$ is nuclear, from the above theorem follows nuclearity of the generalized Pilipović space.

By analogous argument as in Section 2. one can prove theorem which characterize Hermite representation of the elements of the space $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$.

Theorem 3.2. 1. If $f \in \mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ then for some $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d_0$ its Hermite representation $\{a_n\}_{n\in\mathbb{N}_0^d}$ satisfy

(3.2)
$$
|a_n(f)| \le \exp\left[\sum_{k=1}^d M(\theta_k \sqrt{n_k})\right], \quad n = (n_1, \dots n_d)
$$

and f has the representation:

$$
\langle f, \varphi \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d),
$$

where the sequence ${a_n(\varphi)}_{n \in \mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$.

2. Conversely, if a sequence $\{b_n\}_{n\in\mathbb{N}_0^d}$ satisfy that for some $\theta=(\theta_1,...,\theta_d)\in$ \mathbb{R}^d_0 ,

(3.3)
$$
|b_n| \le \exp\left[\sum_{k=1}^d M(\theta_k \sqrt{n_k})\right],
$$

it is the Hermite representation of a unique $f \in \mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ and the Parseval equation holds:

$$
\langle f, \varphi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\varphi), \quad \varphi \in \mathcal{S}^{(M_p)},
$$

where the sequence ${a_n(\varphi)}_{n \in \mathbb{N}_0^d}$ is the Hermite representative of $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$.

4. Kernel Theorem

In [\[7\]](#page-18-8) Komatsu proved, under the assumptions (M.1), (M.2) and (M.3) the kernel theorem for the spaces of ultradistributions $\mathcal{D}^{\{M_p\}'}(\mathbb{R}^d)$ and $\mathcal{D}^{(M_p)'}(\mathbb{R}^d)$ and ultradistributions with compact support, $\mathcal{E}^{\{M_p\}'}(\mathbb{R}^d)$ and $\mathcal{E}^{(M_p)'}(\mathbb{R}^d)$, which are an analogue of L. Schwartz Kernel theorem for distributions.

The kernel theorem for generalized Gelfand-Shilov space states that every continuous linear map K on the space $(\mathcal{S}^{\{M_p\}}(\mathbb{R}^l))_x$ of test functions in some variable x, into the dual space $(\mathcal{S}^{\{M_p\}'}(\mathbb{R}^s))_y$ in a second variable y, is given by a unique element of generalized Gelfand-Shilov space K in both variables x and y .

Using characterizations obtained in Theorems [2.3](#page-7-0) and [2.4](#page-9-1) (resp. Theorems [3.1](#page-11-0) and [3.2\)](#page-12-0), and ideas of B. Simon [\[14\]](#page-18-11), one can give a simple and elegant proof of the kernel theorems for the spaces $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ and $\mathcal{S}^{(M_p)'}(\mathbb{R})$. We will state and prove the kernel theorem only for generalized Gelfand-Shilov spaces. The kernel theorem for generalized Pilipovic space is analogous. Both of the proofs rely heavily on the characterization of Fourier-Hermite coefficients of elements of our spaces and use a minimum amount of real analysis.

Theorem 4.1 (Kernel theorem). Every jointly continuous bilinear functional K on $S^{\{M_p\}}(\mathbb{R}^l) \times S^{\{M_p\}}(\mathbb{R}^s)$ defines a linear map $\mathcal{K}: S^{\{M_p\}}(\mathbb{R}^s) \to$ $S^{\{M_p\}'}(\mathbb{R}^l)$ by

$$
(4.1) \qquad \langle K\varphi,\psi\rangle = K(\psi\otimes\varphi), \qquad \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^s), \psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^l).
$$

and $(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y)$, which is continuous in the sense that $K\varphi_j \to 0$ in $S^{\{M_p\}'}(\mathbb{R}^l)$ if $\varphi_j \to 0$ in $S^{\{M_p\}}(\mathbb{R}^s)$.

Conversely, for linear map $\mathcal{K}: \mathcal{S}^{\{M_p\}}(\mathbb{R}^s) \to \mathcal{S}^{\{M_p\}'}(\mathbb{R}^l)$ there is unique tempered ultradistribution $K \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^{l+s})$ such that (4.1) is valid. The tempered ultradistribution K is called the kernel of K .

Proof. If K is a jointly continuous bilinear functional $\in \mathcal{S}^{\{M_p\}}(\mathbb{R}^l) \times$ $\mathcal{S}^{\{M_p\}}(\mathbb{R}^s)$, then [\(4.1\)](#page-13-0) defines a tempered ultradistribution $(\mathcal{K}\varphi) \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^l)$ since $\psi \mapsto K(\psi \otimes \varphi)$ is continuous. The mapping $\mathcal{K} : \mathcal{S}^{\{M_p\}}(\mathbb{R}^s) \to$ $S^{\{M_p\}'}(\mathbb{R}^l)$ is continuous since the mapping $\varphi \mapsto K(\psi \otimes \varphi)$ is continuous.

Let us prove the converse. To prove the existence we define a bilinear form B on $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^l) \otimes \mathcal{S}^{\{M_p\}'}(\mathbb{R}^s)$ by

$$
B(\varphi, \psi) = \langle K\psi, \phi \rangle, \quad \psi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^l), \varphi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^s).
$$

The form B is a separately continuous bilinear form on the product $\mathcal{S}^{\{M_p\}}(\mathbb{R}^l) \times$ $\mathcal{S}^{\{M_p\}}(\mathbb{R}^s)$ of Frechet spaces and therefore it is jointly continuous, see [\[15\]](#page-18-12).

Let $C > 0, \, \theta \in \mathbb{R}^l_+, \, \nu \in \mathbb{R}^s_+$ be chosen so that

(4.2)
$$
|B(\varphi,\psi)| \leq C ||\varphi||_{\theta} ||\psi||_{\nu},
$$

and let

$$
t_{(n,k)} = B(\mathcal{H}_n, \mathcal{H}_k), \quad n \in \mathbb{N}^l, k \in \mathbb{N}^s.
$$

Since B is jointly continuous on $S^{\{M_p\}}(\mathbb{R}^l) \times S^{\{M_p\}}(\mathbb{R}^s)$, for $\varphi = \sum a_n \mathcal{H}_n$ and $\psi = \sum b_k \mathcal{H}_k$ we have that

$$
B(\varphi, \psi) = \sum t_{(n,k)} a_n b_k.
$$

On the other hand, for $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$ and $(\theta, \nu) \in \mathbb{R}^l \times \mathbb{R}^s$, by [\(4.2\)](#page-13-1) we have α || α || || α |||

$$
|t_{(n,k)}| \leq C ||\mathcal{H}_n||_{\theta} ||\mathcal{H}_k||_{\nu} =
$$

\n
$$
= ||\mathcal{H}_{n_1}||_{\theta_1} ||\mathcal{H}_{n_2}||_{\theta_2} \cdots ||\mathcal{H}_{n_l}||_{\theta_l} ||\mathcal{H}_{k_1}||_{\nu_1} ||\mathcal{H}_{k_2}||_{\nu_2} \cdots ||\mathcal{H}_{k_s}||_{\nu_s} =
$$

\n
$$
= \exp[2 \sum_{i=1}^l M(\theta_i \sqrt{n_i})] \exp[2 \sum_{j=1}^s M(\nu_j \sqrt{k_j})].
$$

Thus, from Theorem [2.3](#page-7-0) it follows that the sequence $\{t_{(n,k)}\}_{(n,k)}$ is a Hermite representation of a tempered ultradistribution $K \in \mathcal{S}^{\{M_p\}'}(\mathbb{R}^l \times \mathbb{R}^s)$. Thus

(4.3)
$$
\langle K, \varphi \rangle = \sum t_{(n,k)} c_{(n,k)},
$$

for $\varphi = \sum c_{(n,k)} \mathcal{H}_{n,k} \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^{l+s}).$

 $\text{If } \varphi = \sum a_n \mathcal{H}_n \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^l) \text{ and } \psi = \sum b_k \mathcal{H}_k \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^s) \text{ then } \varphi \otimes \psi$ has the Hermite representation ${a_n b_k}_{(n,k)}$ and we have that for tempered ultradistribution K defined by (4.3)

$$
K(\varphi \otimes \psi) = \sum_{(n,k)} t_{(n,k)} a_n b_k = B(\varphi, \psi),
$$

so $K = B$. This proves the existence.

The uniqueness follows from the fact that K is completely determined by its Hermite representation ${K, \mathcal{H}_{(n,k)} >}_{(n,k)}$ and the fact that for every $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$

$$
\langle K, \mathcal{H}_{(n,k)} \rangle = \langle K, \mathcal{H}_n \otimes \mathcal{H}_k \rangle = B(\mathcal{H}_n, \mathcal{H}_k) = t_{(n,k)}.
$$

QED

5. Proofs of Lemmas

Let us prove lemmas [2.1](#page-6-0) and [2.2.](#page-6-2)

Lemma 5.1 (Lemma [2.1\)](#page-6-0). a) If conditions $(M.1)$, $(M.2)$ and $(M.3)$ " are satisfied, there exist $C > 0$ and $m_0 > 0$ such that for every $m \leq m_0$

(5.1)
$$
\frac{m^{\alpha+\beta}}{M_{\alpha}M_{\beta}}|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x))| \leq C e^{M(8mH\sqrt{n})}.
$$

b) If conditions (M.1), (M.2) and (M.3)"' are satisfied, for every $m > 0$ there exists $C > 0$ such that the estimate [\(2.3\)](#page-6-1) holds.

Proof. From

$$
\mathcal{F}[\mathcal{H}_n] = \sqrt{2\pi} i^n \mathcal{H}_n
$$
, and $\frac{d^{\alpha}}{dx^{\alpha}} \mathcal{F}[\varphi] = \mathcal{F}[(ix)^{\alpha} \varphi]$

it follows that

$$
\mathcal{H}_n^{(\alpha)} = i^{\alpha - n} \frac{1}{\sqrt{2\pi}} \mathcal{F}[\xi^{\alpha} \mathcal{H}_n] \text{ and } \xi^{2\gamma} \mathcal{F}[\varphi] = \mathcal{F}[(-D^2)^{\gamma} \varphi].
$$

This imply that for an even number $\beta \in \mathbb{N}$ it holds:

(5.2)
\n
$$
(1+x^2)^{\beta/2} \mathcal{H}_n^{(\alpha)}(x) = \frac{i^{\alpha-n}}{\sqrt{2\pi}} \mathcal{F} \Big[\Big(1 - \frac{d^2}{d\xi^2} \Big)^{\beta/2} (\xi^{\alpha} \mathcal{H}_n(\xi) \Big] =
$$
\n
$$
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (1+\xi^2) \Big(1 - \frac{d^2}{d\xi^2} \Big)^{\beta/2} (\xi^{\alpha} \mathcal{H}_n(\xi)) \frac{e^{ix\xi}}{1+\xi^2} d\xi.
$$

From

$$
\xi^\alpha\varphi=2^{-\frac{\alpha}{2}}(L^-+L^+)^{\alpha}\varphi
$$

and

$$
\left(1 - \frac{d^2}{dx^2}\right)^{\gamma} = \left(1 - \frac{1}{2}\left(L^+ - L^+\right)^2\right)^{\gamma}
$$

we obtain that

$$
(1 + \xi^2) \left(1 - \frac{d^2}{d\xi^2} \right)^{\beta/2} \left[\xi^\alpha \mathcal{H}_n(\xi) \right] =
$$

= $2^{-\frac{\alpha}{2}} \left(1 + 2^{-\frac{1}{2}} (L^- + L^+)^2 \right) \left(1 - 2^{-1} (L^- - L^+)^2 \right)^{\beta/2} (L^- + L^+)^\alpha \mathcal{H}_n(\xi) =$
= $2^{-\frac{\alpha}{2}} \sum_{\gamma=0}^{\beta/2} {\binom{\beta/2}{\gamma}} \left(-\frac{1}{2} \right)^\gamma (L^- - L^+)^{2\gamma} (L^- + L^+)^\alpha \mathcal{H}_n(\xi) +$
+ $2^{-\frac{\alpha+1}{2}} \sum_{\gamma=0}^{\beta/2} {\binom{\beta/2}{\gamma}} \left(-\frac{1}{2} \right)^\gamma (L^- + L^+)^\alpha (L^- - L^+)^\alpha (L^- + L^+)^\alpha \mathcal{H}_n(\xi).$

The term

$$
(L^- - L^+)^{2\gamma} (L^- + L^+)^{\alpha} \mathcal{H}_n(\xi).
$$

which appear in the sum on the right hand side of the above equality is a sum of $2^{\alpha+2\gamma}$ terms of the form

$$
L^{\sharp_1} L^{\sharp_2} \cdots L^{\sharp_{2\gamma}} L^{\sharp_{2\gamma+1}} \cdots L^{\sharp_{\alpha+2\gamma}} \mathcal{H}_n(\xi)
$$

where \sharp_j stands for + or -.

In $\binom{\alpha+2\gamma}{j}$ of them L^+ appears exactly j times, $j \in \{0, 1, 2, ..., \alpha + 2\gamma\},\$ and in the case

(5.3)
$$
L^{\sharp_1} \cdots L^{\sharp_{\alpha+2\gamma}} \mathcal{H}_n(\xi) = c_{\sharp_1 \sharp_2 \ldots \sharp_{\alpha+2\gamma}} \mathcal{H}_{n+2j-(\alpha+2\gamma)}(\xi),
$$

where $\mathcal{H}_{-k} := 0$ for $k = 1, 2, ...$ and $C_{\sharp_1\sharp_2... \sharp_{\alpha+2\gamma}}$ is a constant. From $L^-\mathcal{H}_n = \sqrt{n} \mathcal{H}_{n-1}$, and $L^+\mathcal{H}_n = \sqrt{n+1} \mathcal{H}_{n+1}$, it follows that

$$
(5.4) \qquad C_{\sharp_1\sharp_2\ldots\sharp_{\alpha+2\gamma}} \leq C_{--\ldots-++\ldots+} =
$$

$$
= \left(\frac{(n+j)!}{n!}\right)^{1/2} \left(\frac{(n+j)!}{(n+j-(\alpha+2\gamma-j))!}\right)^{1/2} \leq (n+j)^{(\alpha+2\gamma)/2}.
$$

Since $||\mathcal{H}_n||_{L^2} = 1$ we have that

$$
||(L^{-}-L^{+})^{2\gamma}(L^{-}+L^{+})^{\alpha} \mathcal{H}_{n}(\xi)||_{L^{2}} = \sum_{j=0}^{\alpha+2\gamma} {\alpha+2\gamma \choose j} (n+j)^{(2\alpha+2\gamma)/2}.
$$

Analogously one can obtain

$$
||(L^{-}+L^{+})^{2}(L^{-}-L^{+})^{2\gamma}(L^{-}+L^{+})^{\alpha} \mathcal{H}_{n}(\xi)||_{L^{2}} = \sum_{j=0}^{\alpha+2\gamma+2} {\alpha+2\gamma+2 \choose j} (n+j)^{(2\alpha+2\gamma)/2}.
$$

From above it follows that for $\beta \in \mathbb{N}$ even:

$$
(5.5)
$$
\n
$$
\left| (1+x^2)^{\beta/2} \mathcal{H}^{(\alpha)}(\xi) \right| =
$$
\n
$$
= \frac{1}{\sqrt{2\pi}} \int \left| (1+\xi^2) \left(1 - \frac{d^2}{d\xi^2} \right)^{\beta/2} [\xi^{\alpha} \mathcal{H}_n(\xi)] \right| \frac{1}{1+\xi^2} d\xi \le
$$
\n
$$
\leq C \Big[\sum_{\gamma=0}^{\beta/2} \left(\frac{\beta}{\gamma} \right)^{\alpha+2\gamma} \binom{\alpha+2\gamma}{j} (n+j)^{(\alpha+2\gamma)/2} +
$$
\n
$$
+ \sum_{j\geq 0}^{\alpha+2\gamma+2} \binom{\alpha+2\gamma+2}{j} (n+j)^{(\alpha+2\gamma+2)/2} \Big] \le
$$
\n
$$
\leq C(n+\alpha+\beta+2)^{(\alpha+\beta+2)/2} \Big[\sum_{\gamma=0}^{\beta/2} \binom{\beta}{\gamma} \binom{\alpha+2\gamma}{j-0} \binom{\alpha+2\gamma}{j} +
$$
\n
$$
+ \sum_{j=0}^{\alpha+2\gamma+2} \binom{\alpha+2\gamma+2}{j} \Big] \le
$$
\n
$$
\leq C(n+\alpha+\beta+2)^{(\alpha+\beta+2)/2} \Big(\sum_{\gamma=0}^{\beta/2} \binom{\beta}{\gamma} \binom{\alpha+2\gamma+2}{2\gamma+2} + 2^{\alpha+2\gamma+2} \Big) \Big] \le
$$
\n
$$
\leq C 4^{\alpha+\beta} (n+\alpha+\beta+2)^{(\alpha+\beta+2)/2} \leq C 8^{\alpha+\beta} (\max(n,\alpha+\beta+2))^{(\alpha+\beta+2)/2}.
$$

For β odd, we have

$$
|(1+x^2)^{\beta/2}\mathcal{H}_n^{(\alpha)}(x)| \le |(1+x^2)^{(\beta+1)/2}\mathcal{H}_n^{(\alpha)}(x)| \le C 8^{\alpha+\beta}(\max(n,\alpha+\beta+3))^{(\alpha+\beta+3)/2}
$$

From above and (M.2) it holds that for every $\alpha, \beta \in \mathbb{N}_0$

$$
\left| (1+x^2)^{\beta/2} \mathcal{H}_n^{(\alpha)}(x) \right| \le C \frac{M_\alpha M_\beta}{m^{\alpha+\beta}} \frac{(8mH)^{\alpha+\beta+3}(\max(n,\alpha+\beta+3))^{(\alpha+\beta+3)/2}}{M_{\alpha+\beta+3}} \le
$$

$$
\le C \frac{M_\alpha M_\beta}{m^{\alpha+\beta}} \max\left(\exp[M(8mH\sqrt{n})], \sup_k \frac{(8mH)^k k^{k/2}}{M_k} \right).
$$

The above estimation and (M.3)" imply that [\(5.1\)](#page-14-0) holds for all $m \le m_0 =$ $(8HL)^{-1}$.

If moreover $(M.3)$ "' holds then for every $m > 0$ there exists C so that [\(5.1\)](#page-14-0) holds which imply the second part of the theorem. QED

Now we prove Lemma [2.2.](#page-6-2)

Lemma 5.2 (Lemma [2.2\)](#page-6-2). a) If $\varphi \in C^{\infty}$ and $N \in \mathbb{N}$ then

$$
(5.6)\quad (L^-L^+)^N\varphi(x) = 2^N(1+x^2 - \frac{d^2}{dx^2})^N\varphi(x) = \sum_{p=0}^{2N} \sum_{q=0}^{2N-p} c_{p,q}^{(N)} x^p \varphi^{(q)}(x),
$$

where $c_{p,q}^{(N)}$ are constants which satisfy inequality

(5.7)
$$
|c_{p,q}^{(N)}| \le 26^N (2N - q)^{N - \frac{p+q}{2}}.
$$

b) Moreover, if conditions (M.1), (M.2) and $k^{k/2} \subset M_k$ are satisfied for $p, q \in \mathbb{N}, p + q \leq 2N$, then it holds:

(5.8)
$$
|c_{p,q}^{(N)}| \leq 52^N \frac{M_N^2}{M_p M_q}.
$$

Proof. Let us first prove inequality (5.7) by induction. For $N = 1$ the estimation is obvious. Let us suppose that (5.6) and (5.7) hold for some $N \in \mathbb{N}$. Then

$$
2^{N+1}(1+x^2-\frac{d^2}{dx^2})^{N+1}\varphi(x)=\sum_{p=0}^{2N+2}\sum_{q=0}^{2N+2-p}c_{p,q}^{(N+1)}x^p\varphi^{(q)}(x),
$$

where

$$
c_{p,q}^{(N+1)}=2\left(c_{p,q}^{(N)}+c_{p-2,q}^{(N)}-c_{p,q-2}^{(N)}-(p+2)(p+1)c_{p+2,q}^{(N)}-2(p+1)c_{p+1,q-1}^{(N)}\right),
$$

for $p, q \in \mathbb{N}, p, q \leq 2(N+1)$. Constants $c_{k,l}^{(N)}$ are equal to zero if $k+l > 2N$ or k or l are negative. Therefore,

$$
\begin{aligned} &(5.9)\\ |c_{p,q}^{(N+1)}| \le 26^N \cdot 2 \cdot [(2N-q)^{N-\frac{p+q}{2}}+(2N-q)^{N+1-\frac{p+q}{2}}+(2(N+1)-q)^{N-\frac{p+q}{2}}+\\ &\quad +(2N-q)^2(2N-q)^{N-\frac{p+q}{2}-1}+3(2N-q)(2N-q)^{N-\frac{p+q}{2}-1}+\\ &\quad +2(2N-q)^{N-\frac{p+q}{2}-1}+2(2N-q)(2N-q)^{N-\frac{p+q}{2}}+2(2N-q)^{N-\frac{p+q}{2}} \le \\ &\le 26^{N+1}\cdot 2\cdot 13 \cdot (2(N+1)-q)^{N+1-\frac{p+q}{2}}. \end{aligned}
$$

Thus, by induction [\(5.7\)](#page-17-0) holds for every $n \in \mathbb{N}$.

Let us now prove [\(5.8\)](#page-17-1). Using estimation [\(5.7\)](#page-17-0), $p + q \le 2N$, condition

$$
(5.10) \t\t k^{k/2} \le C L^k M_k,
$$

the fact that the Gevrey sequence satisfy condition (M.2) with $H = 2$, $A = 1$, and (M.1) and (M.2) we have that

$$
\begin{aligned} (5.11) \qquad \qquad |c_{p,q}^{(N)}| \leq 26^N (2N-q)^{N-\frac{q}{2}} p^{-\frac{p}{2}} \leq 26^N (2N-q)^{N-\frac{q}{2}} p^{-\frac{p}{2}} \frac{M_{2N-(p+q)}}{M_{2N-(p+q)}} \leq \\ \qquad \leq 26^N \frac{(2N-q)^{N-\frac{q}{2}}}{p^{\frac{p}{2}} (2N-(p+q))^{N-\frac{p+q}{2}}} M_{2N-(p+q)} \leq \\ \qquad \leq 26^N \cdot 2^{N-\frac{q}{2}} \cdot 1 \cdot M_{2N-(p+q)} \frac{M_p M_q}{M_p M_q} \leq 52^N \frac{M_{2N}}{M_p M_q}. \end{aligned}
$$

QED

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